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BERGMAN–TOEPLITZ OPERATORS: RADIAL COMPONENT INFLUENCE*

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We analyze the influence of the radial component of a symbol to spectral, compactness, and Fredholm properties of Toeplitz operators, acting on the Bergman space. We show that there exist *compact* Toeplitz operators whose (radial) symbols are *unbounded* near the unit circle $\partial\mathbb{D}$. Studying this question we give several sufficient, and necessary conditions, as well as the corresponding examples. The essential spectra of Toeplitz operators with pure radial symbols have sufficiently rich structure, and even can be massive .

The C^*-algebras generated by Toeplitz operators with radial symbols are commutative, but the semicommutators $[T_a, T_b) = T_a \cdot T_b - T_{a \cdot b}$ are not compact in general. Moreover for bounded operators T_a and T_b the operator $T_{a \cdot b}$ may not be bounded at all.

1 Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and introduce the space $L_2(\mathbb{D})$ with the usual Lebesgue plane measure $d\mu(z) = dxdy$, $z = x + iy$, and its subspace $\mathcal{A}^2(\mathbb{D})$, the Bergman space, consisting of all functions analytic in $\mathbb{D}$. The Bergman orthogonal projection $B_{\mathbb{D}}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ is given by

$$(B_{\mathbb{D}}\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)\, d\mu(\zeta)}{(1 - z\overline{\zeta})^2}.$$

For a function (symbol) $a(z)$ defined in $\mathbb{D}$, as usual, we will denote by T_a the Toeplitz operator, acting on functions $\varphi \in \mathcal{A}^2(\mathbb{D})$ as follows

$$T_a\varphi = B_{\mathbb{D}}\, a\varphi.$$

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A first rough look of the theory of Toeplitz operators gives an impression that this theory is quite similar to one of Toeplitz operators, acting on the Hardy space on the unit circle $\partial\mathbb{D}$. Indeed, say for bounded symbols with continuous extension onto the boundary $\partial\mathbb{D}$ of the unit disk, all essential information about the compactness and Fredholm properties of these operators is given by the restriction of their symbols $a|_{\partial\mathbb{D}}$ onto the unit circle. For such symbols a Toeplitz operator T_a is compact if and only if $a|_{\partial\mathbb{D}} \equiv 0$, its essential spectrum coincides with the image of $a|_{\partial\mathbb{D}}$; both the commutator $[T_a, T_b] = T_a \cdot T_b - T_b \cdot T_a$ and semicommutator $[T_a, T_b) = T_a \cdot T_b - T_{a \cdot b}$ are compact, and the quotient algebra, generated by these Toeplitz operators, modulo the ideal of compact operators is isomorphic to $C(\partial\mathbb{D})$.

In this context Toeplitz operators with radial symbols $a(r)$ are quite trivial, nothing but compact perturbations of scalar operators, $T_{a(r)} = a(1)I + K$.

At the same time one of the principal differences between the Bergman and Hardy space settings is that in the first case there is an additional direction: "inside the domain". As a consequence symbols with quite a nice behaviour with respect to the circular direction may have very complicated irregular behaviour with respect to the radial direction. In particular this reflects that Toeplitz operators with radial symbols may have, and as we will see, do have interesting and rich structure.

The aim of this paper is to analyze the influence of the radial component of a symbol to spectral, compactness, and Fredholm properties of Toeplitz operators, acting on the Bergman space.

To analyze the impact of the radial component itself we devoted Section 3 to the study of Toeplitz operators having pure radial symbols. Note that this topic is not absolutely new. In [2], studying the Toeplitz operators with *bounded* radial symbols, B. Korenblum and K. Zhu found out their two important properties: the diagonal form of Toeplitz operators with respect to the standard polynomial basis in $\mathcal{A}^2(\mathbb{D})$, and the criterium of compactness of such operators. The methods used in [2] did not permite them to consider *unbounded* symbols, which was left as an open problem. Let us mention as well the papers [3, 4, 5, 6], where the compactness of Toeplitz operators with bounded radial symbols were studied in different settings.

Our approach is based on, and continues the results of [8], permitting to consider both *bounded* and *unbounded* symbols. Therefore in Section 2 we recall necessary facts from [8].

It appears that Toeplitz operators with radial symbols possess many interesting properties. In particular there exist *compact* Toeplitz operators whose (radial) symbols are *unbounded* near the unit circle $\partial\mathbb{D}$. Studying this question we give several sufficient, and necessary conditions, as well as the corresponding examples. It appears that the essential spectra of Toeplitz operators with pure radial symbols have sufficiently rich structure, and even can be massive (i.e., have positive plane measure).

In Section 4 we study C^*-algebras generated by Toeplitz operators. First we consider the C^*-algebra generated by bounded Toeplitz operators with pure radial symbols. This algebra is commutative;, i.e., the commutators $[T_a, T_b]$ are always equal to zero. At the same time the semicommutators $[T_a, T_b)$ are not compact in general. Moreover for bounded operators T_a and T_b, whose symbols are unbounded, the operator $T_{a \cdot b}$ may not be bounded at all. That is, contrary to commonly known cases the set of symbols, for which corresponding Toeplitz operators are bounded, neither forms an algebra (under the pointwise multiplication), nor admits any natural norm.

Finally we consider the C^*-algebra generated by Toeplitz operators whose symbols are continuous in the circular direction and quite arbitrary in the radial one.

2 Preliminaries

Recall here necessary facts from [8]. Passing to polar coordinates in the unit disk we have

$$\begin{aligned} L_2(\mathbb{D}) = L_2(\mathbb{D}, d\mu(z)) &= L_2([0,1), rdr) \otimes L_2([0,2\pi), d\alpha) \\ &= L_2([0,1), rdr) \otimes L_2(S^1, \frac{dt}{it}) = L_2([0,1), rdr) \otimes L_2(S^1), \end{aligned}$$

where $S^1 = \partial\mathbb{D}$ is the unit circle, and

$$\frac{dt}{it} = |dt| = d\alpha$$

is the element of length.

Introduce the unitary operator

$$U_1 = I \otimes \mathcal{F} : \; L_2([0,1), rdr) \otimes L_2(S^1) \longrightarrow L_2([0,1), rdr) \otimes l_2 = l_2(L_2([0,1), rdr)),$$

where the Fourier transform $\mathcal{F} : L_2(S^1) \to l_2$ is given by

$$\mathcal{F} : \; f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t)\, t^{-n} \frac{dt}{it}, \qquad n \in \mathbb{Z},$$

and its inverse $\mathcal{F}^{-1} = \mathcal{F}^* : l_2 \to L_2(S^1)$ is given by

$$\mathcal{F}^{-1} : \; \{c_n\}_{n\in\mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n\, t^n.$$

For each $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ introduce the unitary operator

$$u_n : L_2([0,1), rdr) \longrightarrow L_2([0,1), rdr)$$

by the rule

$$(u_n f)(r) = \frac{1}{\sqrt{n+1}}\, r^{-\frac{n}{n+1}}\, f(r^{\frac{1}{n+1}}),$$

then the inverse operator $u_n^{-1} = u_n^* : L_2([0,1), rdr) \longrightarrow L_2([0,1), rdr)$ is given by

$$(u_n^{-1} f)(r) = \sqrt{n+1}\, r^n\, f(r^{n+1}).$$

Finally, define the unitary operator

$$U_2 : l_2(L_2([0,1), rdr)) \longrightarrow l_2(L_2([0,1), rdr)) = L_2([0,1), rdr) \otimes l_2$$

as follows:

$$U_2 : \{c_n(r)\}_{n\in\mathbb{Z}} \longmapsto \{(u_{|n|}c_n)(r)\}_{n\in\mathbb{Z}}.$$

Let $\ell_0(r) = \sqrt{2}$; we have $\ell_0(r) \in L_2([0,1), rdr)$ and $\|\ell_0(r)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r)$, then the one-dimensional projection P_0 of $L_2([0,1), rdr)$ onto L_0 has the form

$$(P_0 f)(r) = \langle f, \ell_0\rangle \cdot \ell_0 = 2\int_0^1 f(\rho)\,\rho\,d\rho. \tag{2.1}$$

Denote by l_2^+ the subspace of (two-sided) l_2, consisting of all sequences $\{c_n\}_{n\in\mathbb{Z}}$, such that $c_n = 0$ for all $n \in \mathbb{Z}_- = \mathbb{Z}\setminus\mathbb{Z}_+$, and introduce the sequence $\chi_+ = \{\chi_+(n)\}_{n\in\mathbb{Z}} \in l_\infty$, where $\chi_+(n) = 1$ for $n \in \mathbb{Z}_+$, and $\chi_+(n) = 0$ for $n \in \mathbb{Z}_-$. Then the orthogonal projection of l_2 onto l_2^+ has obviously the form $\chi_+ I$.

Theorem 2.1 *The unitary operator $U = U_2U_1$ gives an isometric isomorphism of the space $L_2(\mathbb{D})$ onto $L_2([0,1), rdr)\otimes l_2$ under which*

1. *the Bergman space $\mathcal{A}^2(\mathbb{D})$ is mapped onto $L_0 \otimes l_2^+$,*

$$U : \mathcal{A}^2(\mathbb{D}) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$, generated by $\ell_0(r) = \sqrt{2}$,

2. *the Bergman projection $B_{\mathbb{D}}$ is unitary equivalent to the following one,*

$$U\,B_{\mathbb{D}}\,U^{-1} = P_0 \otimes \chi_+ I,$$

where P_0 is the one-dimensional projection (2.1) of $L_2([0,1), rdr)$ onto L_0.

Introduce the isometric imbedding

$$R_0 : l_2^+ \longrightarrow L_2([0,1), rdr)\otimes l_2$$

by the rule

$$R_0 :\ \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto \ell_0(r)\{\chi_+(n)c_n\}_{n\in\mathbb{Z}}.$$

The adjoint operator $R_0^* : L_2([0,1), rdr)\otimes l_2 \to l_2^+$ is given by

$$R_0^* :\ \{c_n(r)\}_{n\in\mathbb{Z}} \longmapsto \left\{\chi_+(n)\int_0^1 c_n(\rho)\sqrt{2}\,\rho\,d\rho\right\}_{n\in\mathbb{Z}_+}.$$

Now the operator $R = R_0^*U$ maps the space $L_2(\mathbb{D})$ onto l_2^+, and the restriction

$$R|_{\mathcal{A}^2(\mathbb{D})} : \mathcal{A}^2(\mathbb{D}) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^*R_0 : l_2^+ \longrightarrow \mathcal{A}^2(\mathbb{D}) \subset L_2(\mathbb{D})$$

is an isometric isomorphism of l_2^+ onto the subspace $\mathcal{A}^2(\mathbb{D})$ of the space $L_2(\mathbb{D})$.

Remark 2.2 We have

$$\begin{aligned} R\,R^* = I \quad &: \quad l_2^+ \longrightarrow l_2^+ \\ R^*R = B_{\mathbb{D}} \quad &: \quad L_2(\mathbb{D}) \longrightarrow \mathcal{A}^2(\mathbb{D}). \end{aligned}$$

Theorem 2.3 *The isometric isomorphism*

$$R^* = U^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2(\mathbb{D})$$

is given by

$$R^* : \{c_n\}_{n\in\mathbb{Z}_+} \longmapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}_+} \sqrt{2(n+1)}\, c_n\, z^n.$$

Corollary 2.4 *The inverse isomorphism*

$$R : \mathcal{A}^2(\mathbb{D}) \longrightarrow l_2^+$$

is given by

$$R : \varphi(z) \longmapsto \left\{ \frac{\sqrt{2(n+1)}}{\sqrt{2\pi}} \int_{\mathbb{D}} \varphi(z)\, \overline{z}^n \, d\mu(z) \right\}_{n\in\mathbb{Z}_+}.$$

Theorem 2.5 *Let $a = a(r)$ be a measurable function on the segment $[0,1]$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\mathbb{D})$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$, acting on l_2^+. The sequence $\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ is given by*

$$\gamma_a(n) = \int_0^1 a(r^{\frac{1}{2(n+1)}})\, dr, \quad n \in \mathbb{Z}_+. \tag{2.2}$$

Corollary 2.6 *The Toeplitz operator T_a with measurable radial symbol $a = a(r)$ is bounded on $\mathcal{A}^2(\mathbb{D})$ if and only if*

$$\gamma_a = \{\gamma_a(n)\}_{n\in\mathbb{Z}_+} \in l_\infty,$$

and

$$\|T_a\| = \sup_{n\in\mathbb{Z}_+} |\gamma_a(n)|.$$

The Toeplitz operator T_a is compact if and only if $\gamma_a \in c_0$ that is

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

3 Toeplitz operators with radial symbols

To study the Toeplitz operators with radial symbols it is useful first to understand the behaviour of sequences of the type (2.2). We have

$$\gamma_a(n) = \int_0^1 a(r^{\frac{1}{2(n+1)}})\, dr = (n+1) \int_0^1 b(u) u^n \, du, \quad n \in \mathbb{Z}_+,$$

with $b(u) = a(\sqrt{u})$. It is natural to assume that

$$\int_0^1 |b(u)| du < \infty,$$

or equivalently

$$\int_0^1 |a(r)| r dr < \infty.$$

That is the sequence

$$\eta_b(n) = \frac{1}{n+1} \gamma_a(n)$$

forms the sequence of the power momentums of the function $b(u)$.

The following uniqueness result is standard in the momentum theory, and is important for us.

Theorem 3.1 *Let*

$$\eta_b(n_k) = 0, \qquad k \in \mathbb{Z}_+,$$

where $n_k = n_0 + dk$, $n_0 \in \mathbb{Z}_+$, $d \in \mathbb{N}$. *Then* $b(u) = 0$ *almost everywhere.*

PROOF. We have

$$\int_0^1 b(u) u^{n_0} u^{dk} du = 0.$$

Changing the variable $u^d = s$, we obtain

$$\int_0^1 \left[b(s^{1/d}) s^{\frac{n_0+1-d}{d}} \right] s^k ds = 0, \qquad k = 0, 1, 2, \dots$$

Now the function in the square brackets belongs to $L_1(0,1)$, and is orthogonal to all polynomials. Thus this function is equal to zero almost everywhere, so the function $b(u) = 0$ almost everywhere as well. □

Corollary 3.2 *There is no function* $b(u) \in L_1(0,1)$ *for which* $\eta_b(n) \neq 0$ *only at a finite number of points.*

The behaviour of a sequence $\gamma_a(n)$, when $n \to \infty$, is completely determined by the behaviour of a function $a(r)$ (or a function b) in a neighborhood of the point $r = 1$. Given $b \in L_1(0,1)$, introduce the function

$$B(s) = \int_s^1 b(u) du.$$

Theorem 3.3 *If the function $B(s)$ when $s \to 1$ has the form*

$$|B(s)| = O(1-s), \tag{3.1}$$

then

$$\sup_{n\in\mathbb{Z}_+} |\gamma_a(n)| < \infty.$$

If

$$|B(s)| = o(1-s), \tag{3.2}$$

then

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

PROOF. Let $b(u) = a(\sqrt{u}) \in L_1(0,1)$. Integrating by parts we have for $n \geq 1$

$$\gamma_a(n) = (n+1)n \int_0^1 B(s)s^{n-1}ds.$$

Let $\varepsilon = \varepsilon(n) = n^{-2/3}$. Then assuming (3.1) estimate

$$\begin{aligned}
|\gamma_a(n)| &\leq (n+1)n \int_{1-\varepsilon}^1 |B(s)|s^{n-1}ds + (n+1)n \int_0^{1-\varepsilon} |B(s)|s^{n-1}ds \\
&\leq (n+1)n\, c_\varepsilon \int_{1-\varepsilon}^1 (1-s)s^{n-1}ds + (n+1)n(1-\varepsilon)^{n-1} \int_0^1 |B(s)|ds \\
&\leq (n+1)n\, c_\varepsilon \left(\frac{s^n}{n} - \frac{s^{n+1}}{n+1}\right)\Bigg|_{1-\varepsilon}^1 + \text{const}\,(n+1)n \exp((n-1)\ln(1-\varepsilon)) \\
&\leq c_\varepsilon(n+1)n \left(\frac{1}{(n+1)n} + \frac{\exp((n+1)\ln(1-\varepsilon))}{n+1}\right) \\
&+ \text{const}\,(n+1)n(\exp(-(n-1)\varepsilon + (n-1)O(\varepsilon^2))) \\
&\leq c_\varepsilon(1 + n\exp(-(n+1)\varepsilon + O(\varepsilon^2))) + \text{const}\, n^2 \exp(-n^{1/3}) \\
&\leq c_\varepsilon + \text{const}\, n^2 \exp(n^{-1/3}),
\end{aligned}$$

where "const" denotes a quantity uniformly bounded in ε. Having (3.1), the quantity c_ε is uniformly bounded on ε, and thus $\gamma_a \in l_\infty$.

Having (3.2), the quantity c_ε can be chosen in such a way that

$$\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{n\to\infty} c_{\varepsilon(n)} = 0,$$

and thus $\gamma_a \in c_0$. □

In fact Theorem 3.3 says that the behaviour near the boundary of a certain average of symbols, rather than the behaviour of the symbols themselves, is responsable for the boundedness and compactness properties of corresponding Toeplitz operators. That is, in spite of bad behaviour of a symbol, which can be even unbounded near the boundary, the corresponding Toeplitz operator can be bounded and even compact.

Example 1. Let

$$a(r) = (1 - r^2)^{-\beta} \sin(1 - r^2)^{-\alpha}, \tag{3.3}$$

where $\alpha > 0$, and $\beta < 1$. Consider the corresponding function

$$B(v) = \int_v^1 (1 - u)^{-\beta} \sin(1 - u)^{-\alpha} du.$$

Changing variables

$$s = (1 - u)^{-\alpha}, \quad u = 1 - s^{-1/\alpha},$$

we have

$$B(v) = \frac{1}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta} \sin s \, ds, \quad \delta = \frac{1 - \beta}{\alpha} + 1.$$

Integrate by parts twice:

$$\begin{aligned} B(v) &= \frac{1}{\alpha}(\cos(1 - v)^{-\alpha})(1 - v)^{\alpha\delta} + \frac{\delta}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta-1} \cos s \, ds \\ &= \frac{1}{\alpha}(\cos(1 - v)^{-\alpha})(1 - v)^{\alpha\delta} - \frac{\delta}{\alpha}(\sin(1 - v)^{-\alpha})(1 - v)^{\alpha(\delta+1)} \\ &- \frac{\delta(\delta + 1)}{\alpha} \int_{(1-v)^{-\alpha}}^{\infty} s^{-\delta-2} \sin s \, ds. \end{aligned}$$

This implies that

$$B(v) = \frac{\cos(1 - v)^{-\alpha}}{\alpha}(1 - v)^{\alpha-\beta+1} + O((1 - v)^{2\alpha-\beta+1}). \tag{3.4}$$

Thus considering the Toeplitz operator T_a with the radial symbol a of the form (3.3) we have

- for $\alpha \geq \beta$ the sequence $\gamma_a(n)$ is bounded, and thus the Toeplitz operator is bounded on $\mathcal{A}^2(\mathbb{D})$;
- for $\alpha > \beta$ the sequence $\gamma_a(n)$ belongs to c_0 and thus the Toeplitz operator is compact on $\mathcal{A}^2(\mathbb{D})$.

Moreover for $\beta \leq 0$ the symbol (3.3) is bounded, while for $\beta > 0$ the symbol (3.3) is unbounded near the boundary $\partial\mathbb{D}$.

The conditions (3.1) and (3.2) are sufficient for boundedness and compactness of an operator T_a, in general. It is known [2], that for bounded symbols $a(r) \in L_\infty(0, 1)$ the condition (3.2) is necessary and sufficient for compactness of T_a on $\mathcal{A}^2(\mathbb{D})$.

A case when conditions (3.1) and (3.2) are necessary for L_1 symbols describes the following theorem.

Theorem 3.4 *Let $b(u) \in L_1(0, 1)$, and $b(u) \geq 0$ almost everywhere. Then the conditions (3.1) and (3.2) are necessary and sufficient for $\gamma_a \in l_\infty$ and $\gamma_a \in c_0$, respectively.*

PROOF. Let $n = [(1-s)^{-1}]$, then

$$\gamma_a(n) \geq (n+1)\int_s^1 b(u)u^n du \geq \text{const}\,(n+1)\int_s^1 b(u)du = \text{const}\,(n+1)B(s).$$

Thus

$$B(s) \leq \text{const}\,(1-s)\gamma_a(n).$$

□

EXAMPLE 2. Consider the following family of radial symbols

$$a_\alpha(r) = (1-r)^{\alpha-1}, \quad \text{where} \quad \alpha > 0,$$

which scales the (polynomial) growth of symbols near the boundary. We have

$$\gamma_{a_\alpha}(n) = (n+1)\int_0^1 (1-\sqrt{r})^{\alpha-1} r^n dr$$

and

$$B_\alpha(s) = \int_s^1 a_\alpha(\sqrt{r})\,dr = \frac{2}{\alpha}s(1-s)^\alpha + \frac{2}{\alpha(\alpha+1)}(1-s)^{\alpha+1}.$$

By Theorem 3.4 the operator T_{a_α} is bounded if and only if $\alpha \geq 1$, and compact if and only if $\alpha > 1$. That is, in this scale unbounded symbols generate unbounded Toeplitz operators. Moreover, as it will follow from Corollary 3.5, to generate bounded or compact Toeplitz operator its unbounded symbol must necessarily have sufficiently sophisticated oscillating behaviour near the unit circle $\partial\mathbb{D}$.

For a non negative symbol $a(r)$ introduce the function

$$m_a(u) = \inf_{r\in[u,1)} a(r),$$

which is obviously always monotone.

Corollary 3.5 *If* $\lim_{u\to 1} m_a(u) = +\infty$ *(which is equivalent to* $\lim_{r\to 1} a(r) = +\infty$*), then the Toeplitz operator* T_a *is unbounded.*

PROOF. Estimate

$$B(s) = \int_s^1 b(u)du \geq \inf_{r\in[s^2,1]} a(r)\cdot(1-s),$$

then $\lim_{t\to\infty}(B(s)/(1-s)) = +\infty$. Thus according to Theorem 3.4 the operator T_a is unbounded. □

Note that for general L_1 symbols the conditions (3.1) and (3.2) fail to be necessary.

EXAMPLE 3. Let

$$a(r) = -(1-\gamma)(1-r^2)^{-\gamma}\sin(1-r^2)^{-\alpha} + \alpha(1-r^2)^{-\alpha-\gamma}\cos(1-r^2)^{-\alpha}. \tag{3.5}$$

Then

$$B(u) = (1-u)^{1-\gamma} \sin(1-u)^{-\alpha}.$$

Suppose that

$$0 < \gamma < \alpha \tag{3.6}$$

and

$$\alpha + \gamma < 1.$$

Then $b(r)(= a(\sqrt{r})) \in L_1(0,1)$ but conditions (3.1) and (3.2) are not realized. However we can show that operator T_a is bounded and compact in $\mathcal{A}^2(\mathbb{D})$. Indeed

$$\gamma_a(n) = (n+1) \int_0^1 b(r) r^n dr = (n+1)n \int_0^1 B(r) r^{n-1} dr.$$

Integrating by parts once more we have

$$\gamma_a(n) = (n+1)n(n-1) \int_0^1 C(r) r^{n-2} dr,$$

where

$$C(r) = \int_r^1 B(s) ds.$$

Setting $\beta = \gamma - 1$ from (3.4) it follows

$$C(r) = \frac{\cos(1-r)^{-\alpha}}{\alpha} (1-r)^{\alpha-\gamma+2} + O((1-r)^{2\alpha-\gamma+2}).$$

Thus

$$|\gamma_a(n)| \leq \text{const} \cdot n^3 \int_0^1 (1-r)^{\alpha-\gamma+2} r^{n-2} dr.$$

Integrating by parts two times we have

$$|\gamma_a(n)| \leq \text{const} \cdot (n+1) \int_0^1 (1-r)^{\alpha-\gamma} r^n dr,$$

and due to (3.6) from Theorem 3.3 we have

$$\lim_{n\to\infty} \gamma_a(n) = 0.$$

In the above example we have used the following fact: second "antiderivative" of the function $b(r)$ in a neighborhood of the point $r = 1$ has the following asymptotic

$$C(r) = o((1-r)^2).$$

This observation hints the following generalization of Theorem 3.3.

Given $b(u)$, introduce the functions

$$b^{(-j)}(u) = \int_u^1 b^{(-(j-1))}(s) ds, \quad j = 1, 2, \ldots,$$

where $b^{(-0)}(s) = b(s)$.

Note, that $B(u) = b^{(-1)}(u)$, and $C(u) = b^{(-2)}(u)$.

Theorem 3.6 *Let a function $b^{(-j)}(s)$, $j \in \mathbb{N}$, has the following asymptotic, when $s \to 1$,*

$$|b^{(-j)}(s)| = O((1-s)^j),$$

then

$$\sup_{n \in \mathbb{Z}_+} |\gamma_a(n)| < \infty.$$

If

$$|b^{(-j)}(s)| = o((1-s)^j)$$

then

$$\lim_{n \to \infty} \gamma_a(n) = 0.$$

PROOF. The poof is analogous to one of Theorem 3.3. Note, that the function (3.5) of Example 3 satisfies the hypothesis of Theorem 3.6 for $j = 2$. □

Another useful characterization of a sequence $\gamma_a(n)$ gives the following theorem.

Theorem 3.7 *Let $b(u) \in L_1(0,1)$. Then*

$$\lim_{n \to \infty} (\gamma_a(n) - \gamma_a(n+1)) = 0. \tag{3.7}$$

PROOF. Consider

$$\begin{aligned} \gamma_a(n) - \gamma_a(n+1) &= (n+1) \int_0^1 (1-u) u^n b(u) du - \int_0^1 u^{n+1} b(u) du \\ &= I_1(n) + I_2(n). \end{aligned}$$

To estimate the first summand find the point of maximum of the function $s(u) = (1-u)u^n$. This point is obviously $u_0 = 1 - \frac{1}{n+1}$, thus

$$\sup_{u \in [0,1]} s(u) = s(u_0) = \frac{\left(1 - \frac{1}{n+1}\right)^n}{n+1} \leq \frac{\text{const}}{n+1}.$$

Let $\varepsilon = \varepsilon(n) = \frac{1}{\sqrt{n}}$, then

$$\begin{aligned} |I_1(n)| &\leq (n+1) \int_{1-\varepsilon}^1 (1-u) u^n |b(u)| du + (n+1) \int_0^{1-\varepsilon} (1-u) u^n |b(u)| du \\ &\leq \text{const} \int_{1-\varepsilon}^1 |b(u)| du + (n+1)(1-\varepsilon)^n \int_0^1 |b(u)| du. \end{aligned}$$

Now from

$$\lim_{n \to \infty} \int_{1-\varepsilon}^1 |b(u)| du = 0$$

and

$$(n+1) \left(1 - \frac{1}{\sqrt{n}}\right)^n = (n+1) \exp\left(-n \left(\frac{1}{\sqrt{n}} - \frac{1}{2n} + O\left(\frac{1}{n^{3/2}}\right)\right)\right) \to 0$$

it follows that

$$\lim_{n\to\infty} I_1(n) = 0.$$

Analogously splitting the integral $I_2(n)$ on segments $[1-1/\sqrt{n}, 1]$ and $[0, 1-1/\sqrt{n}]$ one can show that

$$\lim_{n\to\infty} I_2(n) = 0,$$

which finishes the proof of the theorem. □

Corollary 3.8 *Let $b(u) \in L_1(0,1)$. Then the set of all limit points of the sequence $\gamma_a(n)$ forms a closed connected subset of $\mathbb{C}$. In particular the sequence $\gamma_a(n)$ can not have a finite or countable set of limit points.*

PROOF. Suppose the set K of limit points is not connected. Then there exist two closed subsets K_1 and K_2 (intersecting K) with a positive distance between them such that $K \subset K_1 \cup K_2$. Without loss of generality we can assume that $\gamma_a(n) \in K_1 \cup K_2$ for each n starting from some N. Thus there exist infinitely many $n_j \in \mathbb{N}$ suth that $\gamma_a(n_j) \in K_1$ but, at that time, either $\gamma_a(n_j+1) \in K_2$, or $\gamma_a(n_j-1) \in K_2$, which contradicts (3.7). □

Corollary 3.9 *The essential spectrum of a bounded Toeplitz operator with a radial symbol is always connected.*

That is, if an l_∞ sequence $\gamma_a(n)$ does not have a limit, then the essential spectrum of the corresponding Toeplitz operator may be either a compact connected curve, or a compact connected subset of $\mathbb{C}$ having positive plain measure. Let us show that both these cases can be realized.

EXAMPLE 4. Unit circle and unit interval.
Let $a_p(r) = \alpha_p(\ln r^{-2})^{ip}$, with $\alpha_p \in \mathbb{C}$, and $p \in \mathbb{R}$. Then

$$\begin{aligned} \gamma_{a_p}(n) &= \alpha_p(n+1)\int_0^1 (\ln u^{-1})^{ip} u^n du \\ &= \alpha_p \int_0^1 (\ln(s^{-1/(n+1)}))^{ip} ds \\ &= (n+1)^{-ip}\left[\alpha_p \int_0^1 (\ln s^{-1})^{ip} ds\right]. \end{aligned}$$

Select now α_p in such a way that the multiple in the square brackets is equal to 1, it is easy to see that $\alpha_p = 1/\Gamma(ip+1)$. Then

$$\gamma_{a_p}(n) = (n+1)^{-ip} = \exp(-ip\ln(n+1)).$$

Thus

$$\mathrm{sp}\, T_{a_p} = \text{ess-sp}\, T_{a_p} = S^1.$$

If $c_p(r) = \operatorname{Im} \alpha_p(\ln r^{-2})^{ip}$, then

$$\gamma_{c_p}(n) = -\sin(p \ln(n+1)), \tag{3.8}$$

and

$$\operatorname{sp} T_{c_p} = \text{ess-sp}\, T_{c_p} = [-1, 1].$$

Example 5. Square.
Let $a(r) = c_1(r) + ic_{\sqrt{2}}(r)$, then by (3.8) we have

$$\gamma_a(n) = -(\sin \ln(n+1) + i \sin \sqrt{2} \ln(n+1)).$$

Since the number $\sqrt{2}$ is irrational, the points $\{\gamma_a(n)\}_{n\in\mathbb{Z}_+}$ form a dense set in the square, and thus

$$\operatorname{sp} T_a = \text{ess-sp}\, T_a = [-1, 1] \times [-1, 1].$$

Example 6. A more complicated curve.
Let $a(r) = c_1(r) + ic_2(r)$, then by (3.8) we have

$$\gamma_a(n) = -(\sin \ln(n+1) + i \sin 2 \ln(n+1)).$$

Now the points of this sequence are located on the curve

$$y^2 - 4x^2 + 4x^4 = 0.$$

On the basis of Examples 4 and 5 we describe a sufficiently wide class of sets in the complex plane, which could be essential spectra of some bounded Toeplitz operators with radial symbols.

Corollary 3.10 *The following statements hold.*

(i) *Let $p(t) = \sum_{j=-m}^{n} c_j t^j$, $c_j \in \mathbb{C}$, be a trigonometric polynomial on unit circle $\mathbb{T}$. Then there exists a symbol $a(r) \in L_\infty(0,1)$ for which the essential spectrum of the operator T_a coincides with the image of $\mathbb{T}$ under this polynomial*

$$p(\mathbb{T}) = \{z \in \mathbb{C} : z = p(t), t \in \mathbb{T}\}. \tag{3.9}$$

(ii) *Let $q(u) = \sum_{j=0}^{n} c_j u^j$, $c_j \in \mathbb{C}$, be a polynomial. Then there exists a radial symbol $a(r) \in L_\infty(0,1)$ such that the essential spectrum of the operator T_a is the following positive plane measure subset of $\mathbb{C}$*

$$q([-1,1] \times [-1,1]) = \{z \in \mathbb{C} : z = q(u), u \in [-1,1] \times [-1,1]\}. \tag{3.10}$$

PROOF. To prove the first statement consider

$$a(r) = \sum_{j=-m}^{n} c_j \alpha_j (\ln r^{-2})^{ij},$$

where the constants α_j are defined as in Example 4. Then due to results of this example we have

$$\gamma_a(n) = p((n+1)^{-i}).$$

But $\overline{\{(n+1)^{-i}\}}_{n\in\mathbb{N}} = \mathbb{T}$, which finishes the proof of the first statement.

Passing to the second statement for a given polynomial q consider

$$q(-\sin\ln(n+1) - i\sin\sqrt{2}\ln(n+1)) =$$

$$= q\left(\frac{(n+1)^{-i} - (n+1)^{i}}{2i} + \frac{(n+1)^{\sqrt{2}i} - (n+1)^{-\sqrt{2}i}}{2}\right) = \sum_{j=1}^{M} d_j (n+1)^{\lambda_j i},$$

where $d_j \in \mathbb{C}$, and the real numbers λ_j are of the form $m_j' + m_j''\sqrt{2}$ for some $m_j', m_j'' \in \mathbb{Z}$. Then for the symbol

$$a(r) = \sum_{j=1}^{M} d_j \alpha_j (\ln(r^{-2}))^{\lambda_j i}$$

we have

$$\gamma_a = q(-\sin\ln(n+1) - i\sin\sqrt{2}\ln(n+1)),$$

and due to Example 5 this gives the required result. □

4 Algebras of Toeplitz operators

We are going to consider now the C^*-algebra generated by bounded Toeplitz operators with radial symbols. First observe, that our class of symbols, as well as, corresponding Toeplitz C^*-algebra will have certain peculiarities. In particular, contrary to commonly known and studied cases (see, for example [9]), the Toeplitz operator algebra is commutative, but the semicommutators $[T_{a_1}, T_{a_2}) = T_{a_1} \cdot T_{a_2} - T_{a_1 \cdot a_2}$ are not compact in general. Moreover, the symbols under study do not form an algebra (under the pointwise multiplication). That is, having two radial symbols $a_1(r)$ and $a_2(r)$, for which the corresponding Toeplitz operators $T_{a_1(r)}$ and $T_{a_2(r)}$ are bounded, the Toeplitz operator $T_{a_1 \cdot a_2}$, which corresponds to the product of these symbols, is not necessarily bounded. The natural permitted structure on the set of symbols under consideration is a linear space (in the algebraic sense, i.e., no norm structure assumed).

EXAMPLE 7. Let

$$a_1(r) = \sin(1-r^2)^{-\alpha}$$

and

$$a_2(r) = (1-r^2)^{-\beta}\sin(1-r^2)^{-\alpha}$$

where $0 \le \beta < 1$ and $\beta < \alpha$. Then according to the Example 1 both operators T_{a_1} and T_{a_2} are bounded and compact.

The product $a_1 \cdot a_2$ has the form

$$a_1(r)\cdot a_2(r) = \frac{(1-r^2)^{-\beta}}{2} - \frac{(1-r^2)^{-\beta}\cos 2(1-r^2)^{-\alpha}}{2}.$$

Let $\beta = 0$, then $T_{a_1\cdot a_2} = \frac{1}{2}I - T_{a_3}$, where the operator T_{a_3} with the symbol

$$a_3(r) = \frac{1}{2}\cos 2(1-r^2)^{-\alpha}$$

is compact. The compactness of T_{a_3} can be shown as in Example 1.

Now let $\beta > 0$. Then the operator T_{a_4} with the symbol

$$a_4(r) = \frac{1}{2}(1-r^2)^{-\beta}$$

according to Example 2 is unbounded . At the same time the operator T_{a_5} with the symbol

$$a_5(r) = \frac{1}{2}(1-r^2)^{-\beta}\cos 2(1-r^2)^{-\alpha}$$

is compact (again analogously to Example 1).

That is

(i) for $\beta = 0$ the operators T_{a_1} and T_{a_2} are compact, the operator $T_{a_1\cdot a_2}$ is bounded but not compact; that is, the semicommutator $[T_{a_1}, T_{a_2}) = T_{a_1}\cdot T_{a_2} - T_{a_1\cdot a_2}$ is *not compact*;

(ii) for $\beta > 0$ the operators T_{a_1} and T_{a_2} are bounded, but the operator $T_{a_1\cdot a_2}$ is *not bounded* at all.

Denote by $\mathcal{M}$ the linear space of measurable functions such that for each $a(r) \in \mathcal{M}$ the Toeplitz operator $T_{a(r)}$ is bounded on $L_2(\mathbb{D})$, and denote by $\mathcal{T}(\mathcal{M})$ the C^*-algebra, which is generated by all Toeplitz operators T_a with symbols $a \in \mathcal{M}$. Let $l(\mathcal{M})$ be the C^*-subalgebra of l_∞ generated by all sequences γ_a for $a \in \mathcal{M}$, that is, $l(\mathcal{M}) = R\,\mathcal{T}(\mathcal{A}_r)R^*$, and let

$$\widehat{l}(\mathcal{M}) = l(\mathcal{M})/(\mathcal{K}\cap l(\mathcal{M})) = l(\mathcal{M})/(c_0\cap l(\mathcal{M})) \quad \subset\ l_\infty/c_0.$$

Theorem 4.1 *The C^*-algebra $\mathcal{T}(\mathcal{M})$ is commutative and isomorphically isometric to the algebra $l(\mathcal{M})$. The isomorphism*

$$\nu : \mathcal{T}(\mathcal{M}) \longrightarrow l(\mathcal{M})$$

is generated by the following mapping

$$\nu : T_a \longmapsto \gamma_a,$$

where $a(r) \in \mathcal{M}$, and the sequence γ_a is given by (2.2).

The Fredholm symbol algebra of the algebra $\mathcal{T}(\mathcal{M})$, i.e. the image of $\mathcal{T}(\mathcal{M})$ in the Calkin algebra:

$$\mathrm{Sym}\,\mathcal{T}(\mathcal{M}) = (\mathcal{T}(\mathcal{M}) + \mathcal{K})/\mathcal{K} = \mathcal{T}(\mathcal{M})/(\mathcal{K} \cap \mathcal{T}(\mathcal{M}),$$

where $\mathcal{K}$ is the ideal of all compact on $\mathcal{A}^2(\mathbb{D})$ operators, is isomorphic and isometric to the algebra $\widehat{l}(\mathcal{M})$. Under their identification the symbol homomorphism

$$\mathrm{sym}\ : \mathcal{T}(\mathcal{M}) \longrightarrow \mathrm{Sym}\,\mathcal{T}(\mathcal{M}) = \widehat{l}(\mathcal{M})$$

is generated by the following mapping

$$\mathrm{sym}\ : T_a \longmapsto \gamma_a + c_0 \cap l(\mathcal{M}) \in \widehat{l}(\mathcal{M}).$$

Proof. Follows directly from Theorem 2.5 and Corollary 2.6. □

Note, that due to the first statement of Corollary 3.10 both the set of invertible operators and the set of Fredholm operators in $\mathcal{T}(\mathcal{M})$ have non trivial homotopic structures.

Introduce now the linear space

$$\mathcal{A} = \mathcal{M} \otimes C(S^1),$$

where the tensor product is understood in the algebraic sense.

We will study the C^*-algebra $\mathcal{T}(\mathcal{A})$, generated by all Toeplitz operators T_a with symbols $a \in \mathcal{A}$. Note that each Toeplitz operator T_a with continuous symbol $a(z) \in C(\overline{\mathbb{D}})$ belongs to $\mathcal{T}(\mathcal{A})$.

Denote by $\mathcal{A}_t = \mathbb{C} \otimes C(S^1)$ the subclass of $\mathcal{A}$ consisting of functions depending only on circular variable t.

Lemma 4.2 *For each function $c = c(t) \in \mathcal{A}_t$ the commutator $[B_{\mathbb{D}}, cI]$ is compact. The Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_t)$ is commutative modulo compact operators, and $\mathcal{T}(\mathcal{A}_t)/\mathcal{K} \cong C(S^1)$. The (Fredholm) symbol homomorphism*

$$\tau : \mathcal{T}(\mathcal{A}_t) \longrightarrow \mathcal{T}(\mathcal{A}_t)/\mathcal{K} \cong C(S^1)$$

is generated by the following mapping of the generators of the algebra $\mathcal{T}(\mathcal{A}_t)$

$$\tau : T_c \longmapsto c(t),$$

where $c(t) \in \mathcal{A}_t$.

Proof. The proof is obvious. □

To describe the (Fredholm) symbol algebra $\mathrm{Sym}\,\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{A})/\mathcal{K} = \widehat{\mathcal{T}(\mathcal{A})}$ of the algebra $\mathcal{T}(\mathcal{A})$ we will use the standard local principle (see, for example, [1], [7]). The algebra $\widehat{\mathcal{T}(\mathcal{A}_t)} = \mathcal{T}(\mathcal{A}_t)/\mathcal{K} \cong C(S^1)$ is obviously a central commutative subalgebra of $\widehat{\mathcal{T}(\mathcal{A})} =$

Sym $\mathcal{T}(\mathcal{A})$. For each $t_0 \in C(S^1)$ denote by J_{t_0} the maximal ideal of $\widehat{\mathcal{T}(\mathcal{A}_t)}$ corresponding to t_0, and by $J(t_0)$ the closed two-sided ideal of the algebra $\widehat{\mathcal{T}(\mathcal{A})}$, generated by J_{t_0}. Then the local algebra at the point t_0 is defined as $\mathcal{T}(t_0) = \widehat{\mathcal{T}(\mathcal{A})}/J(t_0)$, and the natural projection

$$\pi_{t_0} : \mathcal{T}(\mathcal{A}) \longrightarrow \mathrm{Sym}\, \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(t_0)$$

identifies elements of the algebra $\mathcal{T}(\mathcal{A})$ locally equivalent at the point t_0.

Fix a point $t_0 \in S^1$. For each function $a(z) = a(rt) \in \mathcal{A}$ introduce the function $b(r) \in \mathcal{A}_r$ by

$$b(r) = a(rt_0), \qquad r \in [0, 1),$$

this implies that the Toeplitz operator T_a is locally equivalent at the point t_0 to the Toeplitz operator T_b. Now we have obviously

$$\mathcal{T}(t_0) = (\mathcal{T}(\mathcal{M}) + J(t_0))/J(t_0) \cong \mathcal{T}(\mathcal{M})/(J(t_0) \cap \mathcal{T}(\mathcal{M})).$$

Further,

$$J(t_0) \cap \mathcal{T}(\mathcal{M}) = \mathcal{K} \cap \mathcal{T}(\mathcal{M}),$$

thus

$$\mathcal{T}(t_0) \cong \mathcal{T}(\mathcal{M})/(J(t_0) \cap \mathcal{T}(\mathcal{M})) = \mathcal{T}(\mathcal{M})/(\mathcal{K} \cap \mathcal{T}(\mathcal{M})).$$

Thus by Theorem 4.1 we have the following lemma.

Lemma 4.3 *Given $t_0 \in S^1$, the local (symbol) algebra $\mathcal{T}(t_0)$ is isomorphic to the C^*-algebra $\widehat{l}(\mathcal{M})$. Under their identification the homomorphism*

$$\pi_{t_0} : \mathcal{T}(\mathcal{A}) \longrightarrow \mathrm{Sym}\, \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(t_0) \;\cong\; \widehat{l}(\mathcal{M})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A})$:

$$\pi_{t_0} : T_a \longmapsto \widehat{\gamma}_{a(rt_0)},$$

where $\widehat{\gamma}_{a(rt_0)} = \gamma_{a(rt_0)} + c_0 \cap l(\mathcal{M}) \in \widehat{l}(\mathcal{M})$, $a = a(z) \in \mathcal{A}$, and the sequence $\gamma_{a(rt_0)}$ is given by (2.2).

Now pasting together all local algebras we have

Theorem 4.4 *The (Fredholm) symbol algebra* Sym $\mathcal{T}(\mathcal{A})$ *of the Toeplitz operator algebra $\mathcal{T}(\mathcal{A})$ is isomorphic and isometric to the algebra $C(S^1, \widehat{l}(\mathcal{M}))$. Under their identification the symbol homomorphism*

$$\pi_{t_0} : \mathcal{T}(\mathcal{A}) \longrightarrow \mathrm{Sym}\, \mathcal{T}(\mathcal{A}) \;\cong\; C(S^1, \widehat{l}(\mathcal{M}))$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A})$:

$$\mathrm{sym} : T_a \longmapsto \widehat{\gamma}_{a(rt)}, \qquad t \in S^1,$$

where $a = a(z) \in \mathcal{A}$, and for fixed $t \in S^1$ $\widehat{\gamma}_{a(rt)} = \gamma_{a(rt)} + c_0 \cap l(\mathcal{M}) \in \widehat{l}(\mathcal{M})$.

Each element $\widehat{\gamma} = \gamma + c_0 \cap l(\mathcal{M}) \in \widehat{l}(\mathcal{M})$ defines uniquely a compact connected set $b(\widehat{\gamma})$ consisting of all limit points of an (arbitrary) sequence γ, which belongs to the class $\widehat{\gamma}$.

Corollary 4.5 *The essential spectrum of any operator T in the algebra $\mathcal{T}(\mathcal{A})$ is connected and given by*

$$\text{ess-sp}\, T = \bigcup_{t \in S^1} b((\text{sym}\, T)(t)).$$

In particular, for any $a(z) \in \mathcal{A}$ we have

$$\text{ess-sp}\, T_a = \bigcup_{t \in S^1} b(\widehat{\gamma}_{a(rt)}).$$

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