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DYNAMICS OF PROPERTIES OF TOEPLITZ OPERATORS ON THE UPPER HALF-PLANE: PARABOLIC CASE

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Dedicated to the fond memory of Olga Grudskaya, who generously assisted in the preparation of the figures in this paper

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Abstract. We consider Toeplitz operators $T_a^{(\lambda)}$ acting on the weighted Bergman spaces $\mathcal{A}_\lambda^2(\Pi)$, $\lambda \in [0,\infty)$, over the upper half-plane Π, whose symbols depend on $y = \operatorname{Im} z$. Motivated by the Berezin quantization procedure we study the dependence of the properties of such operators on the parameter of the weight λ and, in particular, under the limit procedure $\lambda \to \infty$.

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1. INTRODUCTION

This is the first part of a two-paper set devoted to the study of Toeplitz operators acting on weighted Bergman spaces on the upper half-plane. Both of them are motivated by the same ideas and are a continuation of our research started in [8]. We mentioned in [8] the papers [3], [4], [5], [12], [13], where Toeplitz operators with smooth (or continuous) symbols acting on the weighted Bergman spaces, as well as C^*-algebras generated by such operators, naturally appear in the context of problems in mathematical physics. In particular, recall that given a smooth symbol $a = a(z)$, the family of Toeplitz operators $T_a = \{T_a^{(h)}\}$, with $h \in (0,1)$, is considered under the Berezin quantization procedure ([3], [4]). For a fixed h the Toeplitz operator $T_a^{(h)}$ acts on the weighted Bergman space $\mathcal{A}_h^2$. In the Berezin special quantization procedure ([3], [4]) each Toeplitz operator $T_a^{(h)}$ is represented by its Wick symbol $\widetilde{a}_h$, and the correspondence principle says that for smooth symbols one has

$$\lim_{h\to 0} \widetilde{a}_h = a.$$

Moreover, by [11] the above limit remains valid in the L_1-sense for a wider class of symbols.

As in a quantization procedure, weighted Bergman spaces appear naturally in many questions of complex analysis and operator theory. In the last cases a weight parameter is normally denoted by λ and runs through $(-1, +\infty)$. In the sequel we will consider weighted Bergman spaces $\mathcal{A}^2_\lambda$ parameterized by $\lambda \in (-1, +\infty)$ which is connected with $h \in (0, 1)$, used as the parameter in the quantization procedure, by the rule $\lambda + 2 = \frac{1}{h}$.

At this stage some important problems emerge: *study of the behavior of different properties (boundedness, compactness, spectral properties, etc.) of $T^{(\lambda)}_a$ in dependence on λ, and compare their limit behavior under $\lambda \to \infty$ with corresponding properties of the initial symbol a.*

It seems to be quite impossible to get a reasonably complete answer to the above problem for general (smooth) symbols even for the simplest case of the weighted Bergman spaces on the unit disk (hyperbolic plane). In the same time, the recently discovered classes of commutative $*$-algebras of Toeplitz operators on the unit disk suggest the classes of symbols for which the satisfactory complete answer can be given. Recall in this connection (for details see [15], [16]) that all known cases of commutative $*$-algebras of Toeplitz operators on the unit disk are classified by pencils of (hyperbolic) geodesics of the following three possible types: geodesics intersecting in a single point (elliptic pencil), parallel geodesics (parabolic pencil), and disjoint geodesics, i.e., all geodesics orthogonal to a given one (hyperbolic pencil). Symbols which are constant on the cycles, i.e, on the orthogonal trajectories to the geodesics forming a pencil, generate in each case a commutative $*$-algebra of Toeplitz operators. Moreover, these commutative properties of the Toeplitz operators do not depend at all on smoothness properties of symbols; the symbols can be merely measurable.

The model case for elliptic pencils, Toeplitz operators on the unit disk with radial symbols, has been considered in [8]. In the present paper we consider the model case for parabolic pencils, while the other paper, [9] from this two-paper set is devoted to the study of the model case for hyperbolic pencils. Together, these papers cover the material that remained uncovered after [8]. The results for other (non model) cases can be easily obtained by means of Möbius transformations.

We study Toeplitz operators on the upper half-plane equipped with the hyperbolic metric, where the model case for parabolic pencils is realized as Toeplitz operators with symbols depending only on $y = \operatorname{Im} z$.

The key feature of symbols constant on cycles, which permits us to get much more complete information that one obtained studying general symbols, is as follows. In each case of a commutative $*$-algebra generated by Toeplitz operators, the Toeplitz operators admit a spectral type representation, i.e., they are unitary equivalent to multiplication operators, by a certain sequence in the elliptic case and by certain functions on $\mathbb{R}_+$ and $\mathbb{R}$ in the parabolic and hyperbolic cases, respectively.

We mention a certain difference between the already studied elliptic case [8] and the remaining cases. In particular, in the elliptic case Toeplitz operators have a discrete spectrum and can be compact even for symbols that are unbounded near the boundary, while in both parabolic and hyperbolic cases, Toeplitz operators always have only a continuous spectrum and, being nonzero, can not be compact.

As in the preceding paper, [8], the word "dynamics" in the title stands for the emphasis of our main theme: *what happens to properties of Toeplitz operators acting on weighted Bergman spaces when the weight parameter varies.*

In this paper, as a custom in operator theory, we consider weighted Bergman spaces depending on a real parameter $\lambda \in (-1, \infty)$.

Denote by Π the upper half-plane in $\mathbb{C}$, and introduce the weighted Hilbert space $L_2(\Pi, \mathrm{d}\mu_\lambda)$ which consists of measurable functions f on Π for which the norm

$$\|f\|_{L_2(\Pi,\mathrm{d}\mu_\lambda)} = \Big(\int_\Pi |f(z)|^2 \mathrm{d}\mu_\lambda(z) \Big)^{1/2}$$

is finite. Here $\mathrm{d}\mu_\lambda(z) = \mu_\lambda(z)\mathrm{d}v(z)$ with

$$\mu_\lambda(z) = (\lambda+1)(2\mathrm{Im}\, z)^\lambda, \quad \mathrm{d}v(z) = \frac{1}{\pi}\, \mathrm{d}x\mathrm{d}y, z = x + \mathrm{i}y.$$

Let further $\mathcal{A}^2_\lambda(\Pi)$ denote the weighted Bergman space consisting on functions that are analytic in the upper half-plane Π and belongin to $L_2(\Pi, \mathrm{d}\mu_\lambda)$.

It is well known (see, for example, [13]) that the orthogonal Bergman projection $B_{\Pi,\lambda}$ of $L_2(\Pi, \mathrm{d}\mu_\lambda)$ onto the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ has the form

$$\begin{aligned}(B_{\Pi,\lambda} f)(z) &= (\lambda+1) \int_\Pi f(\zeta) \left(\frac{\zeta - \overline{\zeta}}{z - \overline{\zeta}} \right)^{\lambda+2} \frac{\mathrm{d}v(\zeta)}{(2\,\mathrm{Im}\,\zeta)^2} \\ &= \mathrm{i}^{\lambda+2} \int_\Pi \frac{f(\zeta)}{(z-\overline{\zeta})^{\lambda+2}}\, \mathrm{d}\mu_\lambda(\zeta).\end{aligned}$$

Given a function (symbol) $a = a(z)$, $z \in \Pi$, the Toeplitz operators $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ is defined as follows

$$T_a^{(\lambda)} f = B_{\Pi,\lambda} a f, \quad f \in \mathcal{A}^2_\lambda(\Pi).$$

We start with the description of the Bargmann type transform, the unitary operator which maps the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ onto $L_2(\mathbb{R}_+)$. Besides of its immediate necessity, it provides the unitary equivalence of Toeplitz operators whose symbols depend only on y with the multiplication operators acting on $L_2(\mathbb{R}_+)$, this Bargmann type transform as well as the one established in [9] is of great importance itself and both of them will be used in forthcoming papers in another context.

The key result, which gives an easy access to the properties of Toeplitz operators studied in the paper, is established in Section 2. Namely, we prove that the Toeplitz operator $T_a^{(\lambda)}$ with symbol $a(y)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$ acting on $L_2(\mathbb{R}_+)$, where

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(y/2) y^\lambda \mathrm{e}^{-xy} \mathrm{d}y, \quad x \in \mathbb{R}_+.$$

We mention in this context (see, for example, [3], [5]) the Wick (or covariant, or Berezin) symbol $\widetilde{a}_\lambda(z, \overline{z})$, $z \in \Pi$, of the Toeplitz operator $T_a^{(\lambda)}$, which, together with the so-called star product, carries as well many essential properties of the corresponding Toeplitz operator. Let H be a separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and having a system of coherent states $\{k_g\}_{g\in G}$ parameterized by elements g of some set G carrying a measure (see for details [1], [2]). Recall that the Wick symbol of a bounded linear operator A acting on H is defined as

$$\widetilde{a}_A(g,g) = \frac{\langle Ak_g, k_g\rangle}{\langle k_g, k_g\rangle}, \quad g \in G.$$

In our particular case we have $A = T_a^{(\lambda)}$, $H = \mathcal{A}_\lambda^2(\Pi)$, $G = \Pi$, and $k_g = k_z(\zeta) = \mathrm{i}^{\lambda+2}(\zeta - \overline{z})^{-(\lambda+2)}$, where $z, \zeta \in \Pi$. The star product defines the composition of two Wick symbols $\widetilde{a}_A$ and $\widetilde{a}_B$ of the operators A and B, respectively, as the Wick symbol of the composition AB, i.e., $\widetilde{a}_A \star \widetilde{a}_B = \widetilde{a}_{AB}$.

In Section 3 we give the formulas for the Wick symbols of Toeplitz operators $T_a^{(\lambda)}$, whose symbols depend only on y, as well as the formulas for the star product in terms of our function $\gamma_{a,\lambda}$.

An interesting and important feature of Toeplitz operators on the (weighted) Bergman spaces is that such operators can be *bounded* even for symbols that are *unbounded near the boundary* symbols. In Section 4 we study in details boudedness properties of Toeplitz operators with such unbounded symbols. We give several separate sufficient and necessary boundedness conditions, as well as a number of illustrating examples. It turns out that for unbounded symbols, the behavior of certain means of a symbol, rather than the behavior of a symbol itself, plays a crucial role in the boundedness properties. Given a symbol a, it is natural to introduce the set $B(a)$ of values $\lambda \in [0,\infty)$ for which the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}_\lambda^2(\Pi)$. We show that being nonempty the set $B(a)$ may have only one of the following three types: $[0,\infty)$, $[0,\nu)$, or $[0,\nu]$.

Section 5 is devoted to the spectral properties. The (continuous) spectrum of each $T_a^{(\lambda)}$ coincides with the closure of the image of the corresponding *continuous* function $\gamma_{a,\lambda}$. For each fixed λ the spectrum seems to be quite unrestricted; the definite tendency starts appearing only as λ tends to infinity. The correspondence principle suggests that the limit set of those spectra has to be somehow connected with the range of the initial symbol a. This is definitely true for continuous symbols. Given a continuous symbol a, the limit set of spectra, which we will denote by $M_\infty(a)$, does coincide with the range of a. As in [8], the new effects appear when we consider more complicated symbols. To understand the impact of each type of a discontinuity of a symbol we consider two model cases, piecewise continuous and oscillating symbols.

In the case of piecewise continuous symbols, the limit set $M_\infty(a)$ coincides with the range of a together with the *line segments connecting the one-sided limit points* of our piecewise continuous symbol. Note that these additional line segments may essentially enlarge the limit set $M_\infty(a)$ when compared with the range of a symbol.

In the case of oscillating symbols, the situation becomes more interesting and unexpected. It turns out that in spite of the qualitative identity of symbols, an oscillation type discontinuity, the results may differ drastically depending on a speed of oscillation. We consider two symbols, with strong and, respectively, with slow oscillation. Both of them have the same range, the unit circle, but in the case of strong oscillation, the limit set $M_\infty(a)$ coincides with the *unit disk*, while in the case of slow oscillation $M_\infty(a)$ coincides with the *unit circle*.

For a measurable and, in general, unbounded symbol one always has

$$\operatorname{Range} a \subset M_\infty(a) \subset \operatorname{conv}(\operatorname{Range} a),$$

and the gap between these extreme sets can be substantial. We give a number of examples illustrating possible interrelations between them.

2. Representations of the weighted Bergman space

We start with the description of the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi))$, where $\lambda \in (-1, +\infty)$, which is compatible with the cartesian coordinates in Π. Introduce the unitary operator

$$U_1 = \frac{1}{\sqrt{\pi}}(F \otimes I) : L_2(\Pi, \mathrm{d}\mu_\lambda) \longrightarrow L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, (\lambda+1)(2y)^\lambda \mathrm{d}y),$$

where the Fourier integral transform $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is given by

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} f(x)\, \mathrm{d}x.$$

The image $\mathcal{A}^2_{1,\lambda}(\Pi) = U_1(\mathcal{A}^2_\lambda(\Pi))$ consists of all functions $\varphi = \varphi(x, y)$ satisfying the equation

$$U_1 \frac{\partial}{\partial \overline{z}} U_1^{-1} \varphi = \frac{\mathrm{i}}{2}\left(x + \frac{\partial}{\partial y}\right)\varphi = 0,$$

whose general solution has obviously the form $\varphi(x, y) = \psi(x)\mathrm{e}^{-xy}$. But the function φ has to be in $L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, (\lambda+1)(2y)^\lambda \mathrm{d}y)$, thus $\mathcal{A}^2_{1,\lambda}(\Pi)$ is the set of all functions

$$\varphi(x, y) = \chi_+(x)\theta_\lambda(x) f(x)\mathrm{e}^{-xy}, \quad f \in L_2(\mathbb{R}), \tag{2.1}$$

where $\chi_+(x)$ is the characteristic function of $\mathbb{R}_+$,

$$\theta_\lambda(x) = \Big((\lambda+1)\int\limits_{\mathbb{R}_+} \mathrm{e}^{-2xv}(2v)^\lambda \mathrm{d}v\Big)^{-1/2} = \Big(\frac{2x^{\lambda+1}}{(\lambda+1)\Gamma(\lambda+1)}\Big)^{1/2}, \quad x \geqslant 0, \tag{2.2}$$

and moreover, $\|\varphi\|_{\mathcal{A}^2_{1,\lambda}(\Pi)} = \|f\|_{L_2(\mathbb{R}_+)}$. Introduce the unitary operator

$$U_2 : L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, (\lambda+1)(2y)^\lambda \mathrm{d}y) \longrightarrow L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y)$$

as follows

$$(U_2\varphi)(x, y) = \frac{1}{\theta_\lambda(|x|)} \mathrm{e}^{-y/2 + |x|\beta(|x|, y)} \varphi(x, \beta(|x|, y)),$$

where, for each fixed $x > 0$, the function $\beta(x, y)$ is the inverse function to

$$\gamma(x, t) = -\ln\Big\{\theta^2_\lambda(x)(\lambda+1)\int\limits_t^\infty (2\eta)^\lambda \mathrm{e}^{-2x\eta} \mathrm{d}\eta\Big\}, \tag{2.3}$$

i.e., $\beta(x, \gamma(x,t)) = t,\ x > 0$. We note an alternative form of $\gamma(x,t)$ in terms of the incomplete Γ-function. Start with

$$\int_t^\infty (2\eta)^\lambda \mathrm{e}^{-2x\eta} \mathrm{d}\eta = \frac{1}{2x^{\lambda+1}} \int_{2xt}^\infty u^\lambda \mathrm{e}^{-u} \mathrm{d}u = \frac{1}{2x^{\lambda+1}}\, \Gamma(\lambda+1, 2xt).$$

Then

$$\gamma(x,t) = -\ln\Big\{\Big(\frac{2x^{\lambda+1}}{(\lambda+1)\Gamma(\lambda+1)}\Big)\, \frac{\lambda+1}{2x^{\lambda+1}}\, \Gamma(\lambda+1, 2xt)\Big\} = \ln \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1, 2xt)}.$$

The inverse operator

$$U_2^{-1} : L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y) \longrightarrow L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, (\lambda+1)(2y)^\lambda \mathrm{d}y)$$

has the form $(U_2^{-1}\varphi)(x,y) = \theta_\lambda(|x|)\, \mathrm{e}^{\gamma(|x|,y)/2-|x|y}\varphi(x, \gamma(|x|, y))$. For each $f \in L_2(\mathbb{R})$ one has $U_2 : \chi_+(x)\theta_\lambda(x) f(x)\mathrm{e}^{-xy} \longmapsto \chi_+(x) f(x)\mathrm{e}^{-y/2}$. Thus, the image $\mathcal{A}_2^2 = U_2(\mathcal{A}^2_{1,\lambda}(\Pi))$ is the set of all functions of the form

$$\psi(x,y) = \chi_+(x) f(x)\mathrm{e}^{-y/2}, \quad f \in L_2(\mathbb{R}).$$

We summarize the above in the following theorem.

THEOREM 2.1. *The unitary operator $U = U_2U_1$ gives an isometric isomorphism of $L_2(\Pi, \mathrm{d}\mu_\lambda)$, where $\lambda \in (-1, +\infty)$, onto $L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y)$ and under which:*

(i) *the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$, where L_0 is the one-dimensional subspace of $L_2(\mathbb{R}_+, \mathrm{d}y)$ generated by $l_0(y) = \mathrm{e}^{-y/2}$;*

(ii) *the Bergman projection B^λ_Π is unitary equivalent to*

$$UB^\lambda_\Pi U^{-1} = \chi_+ I \otimes P_0,$$

where P_0 is the one-dimensional projection on L_0

$$(P_0\psi)(y) = \mathrm{e}^{-y/2} \int_0^\infty \psi(v)\mathrm{e}^{-v/2}\mathrm{d}v.$$

Following [14] we introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_0 f)(x,y) = \chi_+(x)\, f(x)\, \ell_0(y).$$

Here the function f is extended to an element of $L_2(\mathbb{R})$ by setting $f(x) \equiv 0$, for $x < 0$. The image of R_0 obviously coincides with the space $\mathcal{A}_2^2$. The adjoint operator $R_0^* : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by $(R_0^*\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,\eta)\, \ell_0(\eta)\, \mathrm{d}\eta$, and $R_0^*R_0 = I : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+)$, $\quad R_0R_0^* = B_2 : L_2(\Pi) \longrightarrow \mathcal{A}_2^2 = L_2(\mathbb{R}_+) \otimes L_0$. Now the operator $R_\lambda = R_0^*U$ maps the space $L_2(\Pi, \mathrm{d}\mu_\lambda)$ onto $L_2(\mathbb{R}_+)$, and the restriction $R_\lambda|_{\mathcal{A}^2_\lambda(\Pi)} : \mathcal{A}^2_\lambda(\Pi) \longrightarrow L_2(\mathbb{R}_+)$ is an isometric isomorphism. The adjoint operator $R_\lambda^* = U^*R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2_\lambda(\Pi) \subset L_2(\Pi, \mathrm{d}\mu_\lambda)$ is an isometric isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2_\lambda(\Pi)$ of the space $L_2(\Pi, \mathrm{d}\mu_\lambda)$.

Remark 2.2. We have

$$R_\lambda R_\lambda^* = I : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \quad R_\lambda^* R_\lambda = B_\Pi^\lambda : L_2(\Pi, \mathrm{d}\mu_\lambda) \longrightarrow \mathcal{A}_\lambda^2(\Pi).$$

Theorem 2.3. *The isometric isomorphism* $R_\lambda^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}_\lambda^2(\Pi)$ *is given by*

$$(R_\lambda^* f)(z) = \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int_{\mathbb{R}_+} f(\xi)\ \xi^{(\lambda+1)/2} \mathrm{e}^{\mathrm{i}z\xi}\ \mathrm{d}\xi. \tag{2.4}$$

Proof. Calculate

$$\begin{aligned}
(R_\lambda^* f)(z) &= (U_1^* U_2^* R_0 f)(z) \\
&= \sqrt{\pi}(F^{-1} \otimes I)(\chi_+(\xi)\, f(\xi)\, \theta_\lambda(\xi)\ \mathrm{e}^{\gamma(\xi,y)/2 - \xi y}\ \mathrm{e}^{-\gamma(\xi,y)/2}) \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \chi_+(\xi)\ f(\xi)\ \frac{\sqrt{2}\ \xi^{(\lambda+1)/2}}{\sqrt{(\lambda+1)\Gamma(\lambda+1)}}\ \mathrm{e}^{-\xi y} \mathrm{e}^{\mathrm{i}x\xi}\ \mathrm{d}\xi \\
&= \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int_{\mathbb{R}_+} f(\xi)\ \xi^{(\lambda+1)/2} \mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)\xi}\ \mathrm{d}\xi. \quad \blacksquare
\end{aligned}$$

Corollary 2.4. *The inverse isomorphism* $R_\lambda : \mathcal{A}_\lambda^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$ *is given by*

$$\begin{aligned}
(R_\lambda \varphi)(x) &= \frac{x^{(\lambda+1)/2}}{\sqrt{\Gamma(\lambda+2)}} \int_\Pi \varphi(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}x}\, \mu_\lambda(w)\, \mathrm{d}v(w) \\
&= \frac{(\lambda+1)\, x^{(\lambda+1)/2}}{\sqrt{\Gamma(\lambda+2)}} \int_\Pi \varphi(\xi + \mathrm{i}\eta)\, \mathrm{e}^{-\mathrm{i}\,(\xi - \mathrm{i}\eta)x}\ (2\eta)^\lambda\ \frac{1}{\pi}\ \mathrm{d}\xi \mathrm{d}\eta.
\end{aligned} \tag{2.5}$$

Let us note that, given a bounded symbol $a = a(z)$, the Toeplitz operator $T_a^{(\lambda)}$ is bounded on all spaces $\mathcal{A}_\lambda^2(\Pi)$, where $\lambda \in (-1, \infty)$, and the corresponding norms are uniformly bounded by $\sup_z |a(z)|$. That is, all spaces $\mathcal{A}_\lambda^2(\Pi)$, where $\lambda \in (-1, \infty)$, are natural and appropriate for Toeplitz operators with *bounded* symbols. One of our aims is a systematic study of *unbounded* symbols. To avoid unnecessary technicalities in what follows we will always assume that $\lambda \in [0, \infty)$.

The above representation of the Bergman space $\mathcal{A}_\lambda^2(\Pi)$ is especially important in the study of the Toeplitz operators with symbols depending only on $y = \operatorname{Im} z$.

Given a function $a = a(y)$ depending only on $y = \operatorname{Im} z$, consider the Toeplitz operator with the symbol $a(y)$

$$T_a^{(\lambda)} : \varphi \in \mathcal{A}_\lambda^2(\Pi) \longmapsto B_{\Pi,\lambda} a \varphi \in \mathcal{A}_\lambda^2(\Pi).$$

In what follows we will, in general, consider *unbounded* symbols. Denote by $L_1(\mathbb{R}_+, 0)$ the class of functions $a(y)$ such that

$$a(y)\mathrm{e}^{-\varepsilon y} \in L_1(\mathbb{R}_+), \quad \text{for any } \varepsilon > 0.$$

THEOREM 2.5. *Given* $a = a(y) \in L_1(\mathbb{R}_+, 0)$, *the Toeplitz operator* $T_a^{(\lambda)}$ *acting on* $\mathcal{A}_\lambda^2(\Pi)$ *is unitary equivalent to the multiplication operator* $\gamma_{a,\lambda} I = R_\lambda T_a^{(\lambda)} R_\lambda^*$, *acting on* $L_2(\mathbb{R}_+)$. *The function* $\gamma_{a,\lambda}(x)$ *is given by*

$$\begin{aligned} \gamma_{a,\lambda}(x) &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2) t^\lambda \mathrm{e}^{-xt} \mathrm{d}t \\ &= \frac{1}{\Gamma(\lambda+1)} \int_0^\infty a(t/2x) t^\lambda \mathrm{e}^{-t} \mathrm{d}t, \quad x \in \mathbb{R}_+. \end{aligned} \tag{2.6}$$

Proof. Calculate

$$\begin{aligned} R_\lambda T_a^{(\lambda)} R_\lambda^* &= R_\lambda B_{\Pi,\lambda} a B_{\Pi,\lambda} R_\lambda^* = R_\lambda (R_\lambda^* R_\lambda) a (R_\lambda^* R_\lambda) R_\lambda^* \\ &= (R_\lambda R_\lambda^*) R_\lambda a R_\lambda^* (R_\lambda R_\lambda^*) = R_\lambda a R_\lambda^* \\ &= R_0^* U_2 U_1 a(y) U_1^{-1} U_2^{-1} R_0 = R_0^* U_2 a(y) U_2^{-1} R_0 = R_0^* a(\beta(|x|, y)) R_0. \end{aligned}$$

Now $(R_0^* a(\beta(|x|, y) R_0 f)(x) = \int_{\mathbb{R}_+} a(\beta(|x|, \eta)\, f(x)\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta = \gamma_{a,\lambda}(x) f(x)$, where for $x \in \mathbb{R}_+$

$$\begin{aligned} \gamma_{a,\lambda}(x) &= \int_{\mathbb{R}_+} a(\beta(|x|, \eta)\, \mathrm{e}^{-\eta}\, \mathrm{d}\eta = \int_{\mathbb{R}_+} a(t) \mathrm{e}^{-\gamma(x,t)} \mathrm{d}\gamma(x,t) \\ &= \int_{\mathbb{R}_+} a(t) \theta_\lambda^2(x) (\lambda+1)(2t)^\lambda \mathrm{e}^{-2tx} \mathrm{d}t = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2) t^\lambda \mathrm{e}^{-xt} \mathrm{d}t. \end{aligned}$$

Here the functions $\gamma(x,t)$ and $\theta_\lambda(x)$ are given by (2.3) and (2.2) respectively. ∎

The above theorem suggests considering not only L_∞-symbols, but unbounded ones as well. It this case we obviously have:

COROLLARY 2.6. *The Toeplitz operator* $T_a^{(\lambda)}$ *with symbol* $a(y)$ *is bounded on* $\mathcal{A}_\lambda^2(\Pi)$ *if and only if the corresponding function* $\gamma_{a,\lambda}(x)$ *is bounded.*

3. TOEPLITZ OPERATORS WITH SYMBOLS DEPENDING ON $y = \operatorname{Im} z$

Reverting the statement of Theorem 2.5 we come to the following spectral-type representation of a Toeplitz operator.

THEOREM 3.1. *Let* $a(y) \in L_1(\mathbb{R}_+, 0)$. *Then the Toeplitz operator* $T_a^{(\lambda)}$ *acting on* $\mathcal{A}_\lambda^2(\Pi)$ *admits the representation*

$$(T_a^{(\lambda)} \varphi)(z) = \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int_{\mathbb{R}_+} t^{(\lambda+1)/2}\, \gamma_{a,\lambda}(t)\, f(t)\, \mathrm{e}^{\mathrm{i}zt} \mathrm{d}t, \tag{3.1}$$

where $f(x) = (R_\lambda \varphi)(x)$.

Proof. Follows directly from Theorems 2.5, 2.3, and Corollary 2.4.

At the same time it is instructive to give a direct proof of the theorem which does not use the results of the previous section. Indeed, for a symbol $a = a(y)$ depending only on y consider the Toeplitz operator

$$(T_a^{(\lambda)}\varphi)(z) = (\lambda+1)\int\limits_{\Pi} a(\eta)\varphi(\zeta)\Big(\frac{\zeta-\overline{\zeta}}{z-\overline{\zeta}}\Big)^{\lambda+2}\frac{\mathrm{d}v(\zeta)}{(2\mathrm{Im}\,\zeta)^2},$$

where $\zeta = \xi + \mathrm{i}\eta$. Represent the function $\varphi(\zeta)$ in the form of the Fourier integral (see (2.1) and (2.2))

$$\varphi(\xi+\mathrm{i}\eta) = \frac{1}{\sqrt{\Gamma(\lambda+2)}}\int\limits_{\mathbb{R}_+} t^{(\lambda+1)/2} f(t)\,\mathrm{e}^{\mathrm{i}t(\xi+\mathrm{i}\eta)}\mathrm{d}t, \quad \eta>0,$$

where $f \in L_2(\mathbb{R}_+)$. Now

$$(T_a^{(\lambda)}\varphi)(z) = \frac{\mathrm{i}^{\lambda+2}(\lambda+1)}{\pi\sqrt{\Gamma(\lambda+2)}}\int\limits_{\mathbb{R}_+} a(\eta)(2\eta)^{\lambda}\mathrm{d}\eta\int\limits_{\mathbb{R}_+} t^{(\lambda+1)/2} f(t)\,\mathrm{e}^{-t\eta}\mathrm{d}t\int\limits_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}t\xi}\,\mathrm{d}\xi}{(z+\mathrm{i}\eta-\xi)^{\lambda+2}}.$$

Using the following formula (see 3.382.6 of [7])

$$\int\limits_{\mathbb{R}}(\mathrm{i}\beta-\xi)^{-(\lambda+2)}\mathrm{e}^{\mathrm{i}t\xi}\mathrm{d}\xi = \chi_+(t)\frac{2\pi}{\mathrm{i}^{\lambda+2}}\,\frac{t^{\lambda+1}\,\mathrm{e}^{-\beta t}}{\Gamma(\lambda+2)}, \tag{3.2}$$

where $\chi_+(t)$ is the characteristic function of $(0,\infty)$, we have

$$\begin{aligned}(T_a^{(\lambda)}\varphi)(z) &= \frac{2(\lambda+1)}{\Gamma(\lambda+2)^{3/2}}\int\limits_{\mathbb{R}_+} a(\eta)(2\eta)^{\lambda}\mathrm{d}\eta\int\limits_{\mathbb{R}_+} t^{(\lambda+1)/2+(\lambda+1)} f(t)\ \mathrm{e}^{-2t\eta+\mathrm{i}zt}\mathrm{d}t\\ &= \frac{2}{\Gamma(\lambda+2)^{1/2}}\int\limits_{\mathbb{R}_+} t^{(\lambda+1)/2} f(t)\ \mathrm{e}^{\mathrm{i}zt}\mathrm{d}t\ \frac{t^{\lambda+1}}{\Gamma(\lambda+1)}\int\limits_{\mathbb{R}_+} a(\eta)(2\eta)^{\lambda}\mathrm{e}^{-2t\eta}\mathrm{d}\eta\\ &= \frac{1}{\sqrt{\Gamma(\lambda+2)}}\int\limits_{\mathbb{R}_+} t^{(\lambda+1)/2} f(t)\ \gamma_{a,\lambda}(t)\ \mathrm{e}^{\mathrm{i}zt}\mathrm{d}t,\end{aligned}$$

where

$$\gamma_{a,\lambda}(t) = \frac{t^{\lambda+1}}{\Gamma(\lambda+1)}\int\limits_0^{\infty} a(\eta/2)\,\eta^{\lambda}\ \mathrm{e}^{-t\eta}\ \mathrm{d}\eta. \quad \blacksquare$$

Theorem 3.2. *Given $a = a(y) \in L_1(\mathbb{R}_+, 0)$, the Wick symbol $\widetilde{a}_\lambda(z,\overline{z})$ of the Toeplitz operator $T_a^{(\lambda)}$ depends only on y as well, and has the form*

$$\widetilde{a}_\lambda(y) = \widetilde{a}_\lambda(z,\overline{z}) = \frac{\langle T_a^{(\lambda)}k_z, k_z\rangle}{\langle k_z, k_z\rangle} = \frac{(2y)^{\lambda+2}}{\Gamma(\lambda+2)}\int\limits_{\mathbb{R}_+} u^{\lambda+1}\,\gamma_{a,\lambda}(u)\,\mathrm{e}^{-2yu}\,\mathrm{d}u, \tag{3.3}$$

and the corresponding Wick function is given by the formula

$$\widetilde{a}_\lambda(z,\overline{w}) = \frac{\langle T_a^{(\lambda)}k_w, k_z\rangle}{\langle k_w, k_z\rangle} = \frac{[-\mathrm{i}(z-\overline{w})]^{\lambda+2}}{\Gamma(\lambda+2)}\int\limits_{\mathbb{R}_+} u^{\lambda+1}\ \gamma_{a,\lambda}(u)\ \mathrm{e}^{\mathrm{i}(z-\overline{w})u}\ \mathrm{d}u. \tag{3.4}$$

Proof. Consider $k_z(w) = \mathrm{i}^{2+\lambda}(w-\overline{z})^{-(\lambda+2)} = \mathrm{i}^{2+\lambda}(u+\mathrm{i}v-x+\mathrm{i}y)^{-(\lambda+2)}$ and calculate

$$\begin{aligned}(U_1k_z)(u,v) &= \frac{\mathrm{i}^{2+\lambda}}{\pi\sqrt{2}}\int\limits_{\mathbb{R}}(\xi+\mathrm{i}v-x+\mathrm{i}y)^{-(\lambda+2)}\mathrm{e}^{-\mathrm{i}\xi u}\mathrm{d}\xi \\ &= \frac{\mathrm{i}^{2+\lambda}}{\pi\sqrt{2}}\int\limits_{\mathbb{R}}(\xi+\mathrm{i}(y+v+\mathrm{i}x))^{-(\lambda+2)}\mathrm{e}^{-\mathrm{i}\xi u}\mathrm{d}\xi.\end{aligned}$$

Using (3.2), we have

$$(U_1k_z)(u,v) = \chi_+(u)\frac{\sqrt{2}u^{\lambda+1}}{\Gamma(\lambda+2)}\ \mathrm{e}^{-u(y+v)-\mathrm{i}ux}.$$

Thus,

$$\begin{aligned}\langle T_a^{(\lambda)}k_z,k_z\rangle &= \langle ak_z,k_z\rangle = \langle U_1ak_z,U_1k_z\rangle = \langle aU_1k_z,U_1k_z\rangle = \\ &= \frac{2}{[\Gamma(\lambda+2)]^2}\int\limits_0^\infty\int\limits_0^\infty a(v)\,u^{2(\lambda+1)}\,\mathrm{e}^{-2u(y+v)}(\lambda+1)\,(2v)^\lambda\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{\Gamma(\lambda+2)}\int\limits_0^\infty u^{\lambda+1}\ \mathrm{e}^{-2yu}\mathrm{d}u\ \frac{2u^{\lambda+1}}{\Gamma(\lambda+1)}\int\limits_0^\infty a(v)(2v)^\lambda\mathrm{e}^{-2uv}\ \mathrm{d}v \\ &= \frac{1}{\Gamma(\lambda+2)}\int\limits_{\mathbb{R}_+} u^{\lambda+1}\ \gamma_{a,\lambda}(u)\,\mathrm{e}^{-2yu}\ \mathrm{d}u.\end{aligned}$$

Thus, we have (3.3). The equality (3.4) follows either from (3.3) by the analytic continuation principle, or can be verified by direct calculations. ∎

REMARK 3.3. Formula (3.3) admits an interesting interpretation. Start with a symbol $a = a(y)$ and the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$, calculate corresponding function $\gamma_{a,\lambda}(x)$, $x > 0$, and consider now the Toeplitz operator $T^{(\lambda+1)}_{\gamma_{a,\lambda}}$ with symbol $\gamma_{a,\lambda}(y)$ acting on $\mathcal{A}^2_{\lambda+1}(\Pi)$. Then the corresponding function $\gamma_{\gamma_{a,\lambda},\lambda+1}$ coincides with the Wick symbol of the initial Toeplitz operator $T_a^{(\lambda)}$, i.e.,

$$\widetilde{a}_\lambda(y) = \widetilde{a}_\lambda(z,\overline{z}) = \gamma_{\gamma_{a,\lambda},\lambda+1}(y).$$

REMARK 3.4. Given a symbol $a = a(y) \in L_1(\mathbb{R}_+,0)$, writing the Toeplitz operator $T_a^{(\lambda)}$ in terms of its Wick symbol (see, for example, [1], [2]) we get the formula (3.1). Indeed

$$
\begin{aligned}
(T_a^{(\lambda)}\varphi)(z) &= \int\limits_{\Pi} \widetilde{a}(z,\overline{w})\, \frac{\varphi(w)\, \mathrm{i}^{\lambda+2}}{(z-\overline{w})^{\lambda+2}}\, \mu_\lambda(w)\, \mathrm{d}v(w) \\
&= \int\limits_{\Pi} \frac{[-\mathrm{i}(z-\overline{w})]^{\lambda+2}}{\Gamma(\lambda+2)} \int\limits_{\mathbb{R}_+} u^{\lambda+1}\, \gamma_{a,\lambda}(u)\, \mathrm{e}^{\mathrm{i}(z-\overline{w})u}\, \mathrm{d}u \\
&\qquad \times \frac{\varphi(w)\, \mathrm{i}^{\lambda+2}}{(z-\overline{w})^{\lambda+2}}\, \mu_\lambda(w)\, \mathrm{d}v(w) \\
&= \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int\limits_{\mathbb{R}_+} u^{(\lambda+1)/2}\, \gamma_{a,\lambda}(u)\; \mathrm{e}^{\mathrm{i}zu}\, \mathrm{d}u \\
&\qquad \times \frac{u^{(\lambda+1)/2}}{\sqrt{\Gamma(\lambda+2)}} \int\limits_{\Pi} \varphi(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}u}\, \mu_\lambda(w)\, \mathrm{d}v(w) \\
&= \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int\limits_{\mathbb{R}_+} u^{(\lambda+1)/2}\, \gamma_{a,\lambda}(u)\, (R_\lambda\varphi)(u)\, \mathrm{e}^{\mathrm{i}zu}\, \mathrm{d}u.
\end{aligned}
$$

COROLLARY 3.5. *Let $T_a^{(\lambda)}$ and $T_b^{(\lambda)}$ be two Toeplitz operators with symbols $a(y)$ and $b(y)$ respectively, $a(y), b(y) \in L_1(\mathbb{R}_+, 0)$, and let $\widetilde{a}_\lambda(y)$ and $\widetilde{b}_\lambda(y)$ be their Wick symbols. Then the Wick symbol $\widetilde{c}_\lambda(y)$ of the composition $T_a^{(\lambda)}T_b^{(\lambda)}$ is given by*

$$
\widetilde{c}_\lambda(y) = (\widetilde{a}_\lambda \star \widetilde{b}_\lambda)(y) = \frac{(2y)^{\lambda+2}}{\Gamma(\lambda+2)} \int\limits_{\mathbb{R}_+} u^{\lambda+1}\, \gamma_{a_1,\lambda}(u)\, \gamma_{a_2,\lambda}(u)\, \mathrm{e}^{-2yu}\, \mathrm{d}u.
$$

Proof. Besides of a direct verification based on the formula for the star product of Wick symbols ([3], [4]), the result follows immediately from Theorems 2.5 and 3.2. ∎

4. BOUNDEDNESS OF TOEPLITZ OPERATORS WITH SYMBOLS DEPENDING ON $y = \operatorname{Im} z$

Recall, see Corollary 2.6, that the function

$$
(4.1)\quad \gamma_{a,\lambda}(t) = \frac{t^{\lambda+1}}{\Gamma(\lambda+1)} \int\limits_0^\infty a(\eta/2)\, \eta^\lambda\; \mathrm{e}^{-t\eta}\; \mathrm{d}\eta = \frac{1}{\Gamma(\lambda+1)} \int\limits_0^\infty a(\eta/2t)\, \eta^\lambda\; \mathrm{e}^{-\eta}\; \mathrm{d}\eta
$$

is responsible for the boundedness of a Toeplitz operator with symbol $a = a(y)$. If the symbol $a = a(y) \in L_\infty(\mathbb{R}_+)$, then the operator $T_a^{(\lambda)}$ is obviously bounded on $\mathcal{A}_\lambda^2(\Pi)$, and $\|T_a^{(\lambda)}\| \leqslant \text{ess-sup}|a(y)|$. As it is easy to see, the major contribution to the integral (4.1) for "very big t" , $t \to \infty$, is determined by the values of $a(y)$ at a neighborhood of the point 0, and the major contribution for "very small t",

$t \to 0$, is determined by values of $a(y)$ at a neighborhood of ∞. In particular, if $a(y)$ has limits at the points 0 and ∞, then

$$\lim_{t\to\infty} \gamma_{a,\lambda}(t) = \lim_{y\to 0} a(y),$$
$$\lim_{t\to 0} \gamma_{a,\lambda}(t) = \lim_{y\to\infty} a(y).$$

As a matter of fact, 0 and ∞ are the only points of the unbounded symbols $a(y) \in L_1(\mathbb{R}_+, 0)$ that we have to worry for. Moreover, it is the behavior of certain means of the symbol rather than the behavior of the symbol itself, that plays the crucial role in the study of the boundedness of Toeplitz operators.

Given $\lambda \in [0, +\infty)$ and a locally summable function $a(y)$, we introduce the following means

$$B^{(1)}_{a,\lambda}(\xi) = \int_0^\xi a(t/2)t^\lambda \mathrm{d}t,$$
$$B^{(j)}_{a,\lambda}(\xi) = \int_0^\xi B^{(j-1)}_{a,\lambda}(t)\mathrm{d}t, \quad j = 2, 3, \ldots.$$

THEOREM 4.1. *Let $a(y) \in L_1(\mathbb{R}_+, 0)$. If, for any $\lambda_0 \in [0, +\infty)$ and any $j \in \mathbb{N}$, the function $B^{(j)}_{a,\lambda_0}(\xi)$ has the following asymptotic behaviors in the neighborhoods of the points $\xi = 0$ and $\xi = \infty$*

$$B^{(j)}_{a,\lambda_0}(\xi) = O(\xi^{j+\lambda_0}), \quad \xi \to 0, \tag{4.2}$$

and

$$B^{(j)}_{a,\lambda_0}(\xi) = O(\xi^{j+\lambda_0}), \quad \xi \to \infty, \tag{4.3}$$

then for each $\lambda \in [\lambda_0, \infty)$

$$\sup_{x\in\mathbb{R}_+} |\gamma_{a,\lambda}(x)| < \infty,$$

and the corresponding Toeplitz operator $T^{(\lambda)}_a$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ for each $\lambda \in [\lambda_0, \infty)$.

Proof. Let $\lambda \geqslant \lambda_0$. Assume first that $j = 1$. Then the conditions (4.2) and (4.3) imply that, for all $\xi \in \mathbb{R}_+$, the following estimate holds

$$|B^{(1)}_{a,\lambda_0}(\xi)| \leqslant \mathrm{const}\, \xi^{1+\lambda_0}, \tag{4.4}$$

where "const" does not depend on $\xi \in \mathbb{R}_+$. Integrating by parts we have, for all $x \in \mathbb{R}_+$,

$$\begin{aligned}|\gamma_{a,\lambda}(x)| &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\Big|\int_0^\infty t^{\lambda-\lambda_0}\mathrm{e}^{-xt}\mathrm{d}B^{(1)}_{a,\lambda_0}(t)\Big| \\ &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\Big|\int_0^\infty B^{(1)}_{a,\lambda_0}(t)[(\lambda-\lambda_0)t^{\lambda-\lambda_0-1} - xt^{\lambda-\lambda_0}]\mathrm{e}^{-xt}\mathrm{d}t\Big| \\ &\leqslant \text{const}\,\frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty ((\lambda-\lambda_0)t^{\lambda} + xt^{\lambda+1})\mathrm{e}^{-xt}\mathrm{d}t \\ &\leqslant \text{const}\,[(\lambda-\lambda_0)+(\lambda+1)] = \text{const}\,(2\lambda-\lambda_0+1),\end{aligned}$$

and the case $j=1$ is done.

For $j \geqslant 2$ we use the inequalities

$$|B^{(j)}_{a,\lambda_0}(\xi)| \leqslant \text{const}\,\xi^{j+\lambda_0} \tag{4.5}$$

(where $\xi \in \mathbb{R}_+$ and "const" does not depend on ξ) and integrate by parts j-times. ∎

Remark 4.2. The condition (4.2) provides the boundedness of the function $\gamma_{a,\lambda}(x)$ at a neighborhood of $x=\infty$, while the condition (4.3) provides the boundedness of the functions $\gamma_{a,\lambda}(x)$ at a neighborhood of $x=0$.

The next statement sets a partial order on the family of sufficient conditions for boundedness of Toeplitz operators given by Theorem 4.1.

Theorem 4.3. (i) *Let the conditions* (4.2) *and* (4.3) *hold for* $j=j_0$ *and some* λ_0. *Then these conditions hold for* $j=j_0+1$ *and the same* λ_0.

(ii) *Let the conditions* (4.2) *and* (4.3) *hold for* $j=j_0$ *and some* λ_0. *Then these conditions hold for* $j=j_0$ *and* λ_0 *replaced by any* $\lambda_1 \geqslant \lambda_0$.

Proof. Assume we have (4.2) and (4.3) for $j=j_0$. Then, according to (4.5), we have

$$|B^{(j_0+1)}_{a,\lambda_0}(\xi)| \leqslant \int_0^\xi |B^{(j_0)}_{a,\lambda_0}(t)|\mathrm{d}t \leqslant \text{const}\int_0^\xi t^{j_0+\lambda_0}\mathrm{d}t \leqslant \text{const}\,\xi^{j_0+1+\lambda_0}.$$

Thus, the first statement is proved. Let us now have (4.2) and (4.3) for $j=1$ and $\lambda=\lambda_0$. If $\lambda_1>\lambda_0$ then

$$\begin{aligned}|B^{(1)}_{a,\lambda_1}(\xi)| &\leqslant \Big|\int_0^\xi t^{\lambda_1-\lambda_0}\mathrm{d}B^{(1)}_{a,\lambda_0}(t)\Big| \\ &= \Big|B^{(1)}_{a,\lambda_0}(\xi)\xi^{\lambda_1-\lambda_0} - (\lambda_1-\lambda_0)\int_0^\xi B^{(1)}_{a,\lambda_0}(t)t^{\lambda_1-\lambda_0-1}\mathrm{d}t\Big| \\ &\leqslant \text{const}\,\Big(\Big|\xi^{1+\lambda_0}\xi^{\lambda_1-\lambda_0} + \int_0^\xi t^{1+\lambda_0}t^{\lambda_1-\lambda_0-1}\mathrm{d}t\Big|\Big) \leqslant \text{const}\,\xi^{1+\lambda_1}.\end{aligned}$$

Let now (4.2) and (4.3) hold for $j = 2$ and $\lambda = \lambda_0$. Then, for each $\lambda_1 > \lambda_0$, we have

$$
\begin{aligned}
|B^{(2)}_{a,\lambda_1}(\xi)| &= \Big|\int_0^\xi \int_0^u a(t/2)t^{\lambda_1}\mathrm{d}t\mathrm{d}u\Big| = \Big|\int_0^\xi \int_0^u t^{\lambda_1-\lambda_0}\mathrm{d}B^{(1)}_{a,\lambda_0}(t)\mathrm{d}u\Big| \\
&= \Big|\int_0^\xi B^{(1)}_{a,\lambda_0}(u)u^{\lambda_1-\lambda_0}\mathrm{d}u - (\lambda_1-\lambda_0)\int_0^\xi \int_0^u B^{(1)}_{a,\lambda_0}(t)t^{\lambda_1-\lambda_0-1}\mathrm{d}t\mathrm{d}u\Big| \\
&= \Big|B^{(2)}_{a,\lambda_0}(\xi)\xi^{\lambda_1-\lambda_0} - (\lambda_1-\lambda_0)\int_0^\xi B^{(2)}_{a,\lambda_0}(u)u^{\lambda_1-\lambda_0-1}\mathrm{d}u \\
&\quad - (\lambda_1-\lambda_0)\int_0^\xi B^{(2)}_{a,\lambda_0}(u)u^{\lambda_1-\lambda_0-1}\mathrm{d}u \\
&\quad + (\lambda_1-\lambda_0)(\lambda_1-\lambda_0-1)\int_0^\xi \int_0^u B^{(2)}_{a,\lambda_0}(t)t^{\lambda_1-\lambda_0-2}\mathrm{d}t\mathrm{d}u\Big| \\
&\leqslant \mathrm{const}\left(\xi^{2+\lambda_1} + \frac{2(\lambda_1-\lambda_0)}{\lambda_1+2}\xi^{2+\lambda_1} + \frac{(\lambda_1-\lambda_0)(\lambda_1-\lambda_0-1)}{(\lambda_1+1)(\lambda_1+2)}\xi^{2+\lambda_1}\right) \\
&\leqslant \mathrm{const}\,\xi^{2+\lambda_1}.
\end{aligned}
$$

The cases $j_0 > 2$ for the second statement are considered analogously. ∎

EXAMPLE 4.4. Consider the unbounded symbol $a(t/2) = t^{-\beta}\sin t^{-\alpha}$, where $0 < \beta < 1$, $\alpha > 0$. Applying Theorem 4.1 for $j = 1$ and $\lambda_0 = 0$ we have

$$
B_{a,0}(\xi) = \int_0^\xi t^{-\beta}\sin t^{-\alpha}\mathrm{d}t = \frac{1}{\alpha}\int_{\xi^{-\alpha}}^\infty y^{(\beta-1)/\alpha-1}\sin y\,\mathrm{d}y. \tag{4.6}
$$

Integrating by parts two times we get

$$
\begin{aligned}
B^{(1)}_{a,0}(\xi) &= \frac{\xi^{\alpha-\beta+1}}{\alpha}\cos\xi^{-\alpha} - \frac{(\beta-\alpha-1)}{\alpha^2}\xi^{2\alpha-\beta+1}\sin\xi^{-\alpha} \\
&\quad - \frac{(\beta-\alpha-1)(\beta-2\alpha-1)}{\alpha^3}\int_{\xi^{-\alpha}}^\infty y^{(\beta-1)/\alpha-3}\sin y\ \mathrm{d}y.
\end{aligned}
$$

So we have

$$
B^{(1)}_{a,0}(\xi) = \frac{\xi^{\alpha-\beta+1}}{\alpha}\cos\xi^{-\alpha} + O(\xi^{2\alpha-\beta+1}), \quad \xi \to 0. \tag{4.7}
$$

To get the asymptotic at the infinity we use again the representation (4.6):

$$
B^{(1)}_{a,0}(\xi) = \frac{1}{\alpha}\int_{\xi^{-\alpha}}^1 y^{(\beta-\alpha-1)/\alpha}\sin y\ \mathrm{d}y + \frac{1}{\alpha}\int_1^\infty y^{(\beta-\alpha-1)/\alpha}\sin y\ \mathrm{d}y.
$$

Since $((\beta-\alpha-1)/\alpha)<0$ the second integral converges. Integrating by parts the first integral we get

$$B^{(1)}_{a,0}(\xi)=c_0+c_1\xi^{1-\beta-\alpha}+O(\xi^{1-\beta-2\alpha}),\quad c_0,c_1\in\mathbb{C}. \tag{4.8}$$

Thus, if

$$\alpha\geqslant\beta, \tag{4.9}$$

then the conditions (4.2) and (4.3) hold for $j=1$, $\lambda_0=0$, and the operator $T^{(\lambda)}_a$ is bounded for each $\lambda\geqslant 0$.

Now apply Theorem 4.1 for $j=2$, $\lambda_0=0$. Let $\alpha<\beta$. Using the inequality (4.7), for $\beta:=\beta-\alpha-1$, and (4.8) we get $B^{(2)}_{a,0}(\xi)=O(\xi^{2\alpha-\beta+2}),\xi\to 0$, and $B^{(2)}_{a,0}(\xi)=O(\xi)+O(\xi^{2-\beta+\alpha}),\xi\to\infty$. Thus, the operator $T^{(\lambda)}_a$ is bounded if

$$\alpha\geqslant\frac{\beta}{2}. \tag{4.10}$$

Analogously, applying Theorem 4.1 for $j=3,4,\dots$ and $\lambda_0=0$, we have that, for

$$\alpha\geqslant\frac{\beta}{j}, \tag{4.11}$$

operator $T^{(\lambda)}_a$ is bounded. Since there exist j large enough for which (4.11) holds we have that, for arbitrary $0<\beta<1$ and $\alpha>0$, the operator $T^{(\lambda)}_a$ is bounded for each $\lambda\geqslant 0$.

REMARK 4.5. Example 4.4 shows that the conditions (4.2) and (4.3) for $j=j_1$, $\lambda_0=0$, compared with those for $j=j_2$, $\lambda_0=0$, and $j_1>j_2$, widen in fact a class of symbols for which the boundedness of the corresponding Toeplitz operators can be justified.

The sufficient conditions of Theorem 4.1 provide at once the simultaneous boundedness of an operator $T^{(\lambda)}_a$ for all $\lambda\in[\lambda_0,\infty)$. We pass now to a more delicate question concerning the boundedness of a Toeplitz operator $T^{(\lambda)}_a$ on the space $\mathcal{A}^2_\lambda(\Pi)$ with respect to its dependence on λ. The following result plays a central role here.

THEOREM 4.6. *Let $a(y)$ belong to $L_1(\mathbb{R}_+,0)$ and let the operator $T^{(\lambda_0)}_a$ be bounded on $\mathcal{A}^2_{\lambda_0}(\Pi)$ for a certain $\lambda_0>0$. Then $T^{(\lambda)}_a$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ for each $\lambda\in[0,\lambda_0]$.*

Proof. Let the operator $T^{(\lambda)}_a$ be bounded on $\mathcal{A}^2_{\lambda_0}(\Pi)$, that is, $\sup\limits_{x>0}|\gamma_{a,\lambda_0}(x)|$

$< \infty$. Write, for $\lambda < \lambda_0$

$$
\begin{aligned}
\gamma_{a,\lambda}(x) &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2)t^\lambda \mathrm{e}^{-xt}\mathrm{d}t \\
&= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)\Gamma(\lambda_0-\lambda)} \int_0^\infty a(t/2)t^{\lambda_0}\mathrm{e}^{-xt}\mathrm{d}t \int_0^\infty y^{\lambda_0-\lambda-1}\mathrm{e}^{-yt}\mathrm{d}y \\
&= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)\Gamma(\lambda_0-\lambda)} \int_0^\infty y^{\lambda_0-\lambda-1}\mathrm{d}y \int_0^\infty a(t/2)t^{\lambda_0}\mathrm{e}^{-(x+y)t}\mathrm{d}t \\
&= \frac{\Gamma(1+\lambda_0)}{\Gamma(\lambda+1)\Gamma(\lambda_0-\lambda)} \int_0^\infty y^{\lambda_0-\lambda-1}(1+y)^{-\lambda_0-1}\gamma_{a,\lambda_0}(x(1+y))\mathrm{d}y.
\end{aligned}
$$

Thus, we have

$$
|\gamma_{a,\lambda}(x)| \leqslant \sup_{x>0} |\gamma_{a,\lambda_0}(x)| \frac{\Gamma(1+\lambda_0)}{\Gamma(\lambda+1)\Gamma(\lambda_0-\lambda)} \int_0^\infty y^{\lambda_0-\lambda-1}(1+y)^{-\lambda_0-1}\mathrm{d}y. \quad \blacksquare
$$

The next theorem extends the range of λ, given by Theorem 4.1, for simultaneous boundedness of Toeplitz operators on $\mathcal{A}^2_\lambda(\Pi)$.

THEOREM 4.7. *Under the hypothesis of Theorem* 4.1, *the Toeplitz operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$, *for each* $\lambda \in [0,\infty)$.

Proof. Follows directly from Theorems 4.1 and 4.6. $\blacksquare$

Theorem 4.6 allows us to obtain in particular the necessity of the hypothesis of Theorem 4.1 in the case of nonnegative symbols or nonnegative means.

THEOREM 4.8. (i) *Assume that* $a(y) \in L_1(\mathbb{R}_+, 0)$ *and* $a(y) \geqslant 0$ *almost everywhere. Let the operator* $T_a^{(\lambda')}$ *be bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' \geqslant 0$. *Then the conditions* (4.2) *and* (4.3) *hold, for* $j = 1$ *and* $\lambda_0 = 0$, *and, consequently, the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$, *for each* $\lambda \in [0,\infty)$.

(ii) *Assume that* $B^{(j)}_{a,\mu}(y) \geqslant 0$ *almost everywhere for some* $j = j_0 \geqslant 1$ *and* $\mu \geqslant 0$, *and that the operator* $T_a^{(\lambda')}$ *is bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' \geqslant 0$. *Then the conditions* (4.2) *and* (4.3) *hold for* $j = j_0 + 1$ *and* $\lambda_0 = \mu$ *and consequently, the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for each* $\lambda \in [0,\infty)$.

Proof. (i) If $T_a^{(\lambda')}$ is bounded on $\mathcal{A}^2_{\lambda'}(\Pi)$, then according to Theorem 4.6 the operator $T_a^{(0)}$ is bounded on $\mathcal{A}^2_0(\Pi)$. We have

$$
\gamma_{a,0}(x) = x\int_0^\infty a(t/2)\mathrm{e}^{-xt}\mathrm{d}t \geqslant x\int_0^{x^{-1}} a(t/2)\mathrm{e}^{-xt}\mathrm{d}t \geqslant \frac{x}{\mathrm{e}} B^{(1)}_{a,0}(x^{-1}).
$$

Thus, denoting $\xi = x^{-1}$, we have

$$B^{(1)}_{a,0}(\xi) \leqslant \Big(\mathrm{e} \sup_{x\in\mathbb{R}_+} |\gamma_{a,0}(x)|\Big)\xi.$$

(ii) Assume first that $j_0 = 1$ and $\mu \geqslant \lambda'$. We have

$$\gamma_{a,\lambda'}(x) = \frac{x^{\lambda'+1}}{\Gamma(\lambda'+1)} \int_0^\infty a(t/2)t^{\lambda'}\mathrm{e}^{-xt}\,\mathrm{d}t.$$

Integrating by parts we get

$$\begin{aligned}\gamma_{a,\lambda'}(x) &= \frac{x^{\lambda'+1}}{\Gamma(\lambda'+1)} \int_0^\infty B^{(1)}_{a,\mu}(t)[(\mu-\lambda')+xt]t^{\lambda'-\mu-1}\mathrm{e}^{-xt}\mathrm{d}t \\ &\geqslant \frac{x^{\lambda'+1}}{\Gamma(\lambda'+1)}\Big(\int_0^{x^{-1}} B^{(1)}_{a,\mu}(t)\mathrm{d}t\Big)[(\mu-\lambda')+1]x^{-(\lambda'-\mu-1)}\mathrm{e}^{-1} \\ &= \frac{x^{\mu+2}(\mu-\lambda'+1)}{\mathrm{e}\,\Gamma(\lambda'+1)}\, B^{(2)}_{a,\mu}(x^{-1}).\end{aligned}$$

Again, denoting $\xi = x^{-1}$ we have

$$B^{(2)}_{a,\mu}(\xi) \leqslant \frac{\mathrm{e}\,\Gamma(\lambda'+1)}{\mu-\lambda'+1} \sup_{x\in\mathbb{R}_+} |\gamma_{a,\lambda'}(x)|\,\xi^{\mu+2}.$$

The above integration by parts is correct because, for arbitrary $a(t) \in L_1(\mathbb{R}_+, 0)$, we have $|B^{(1)}_{a,\mu}(\xi)| = o(\xi^\mu), \quad \xi \to 0$.

Let now $j_0 = 1$ and $\mu < \lambda'$. Then, according to Theorem 4.6 the operator $T^{(\mu)}_a$ is bounded on $\mathcal{A}^2_\mu(\Pi)$. Repeating the above reasonings for the function $\gamma_{a,\mu}(x)$ we complete the consideration for the case $j_0 = 1$.

The cases $j_0 > 1$ are considered analogously. ∎

Remark 4.9. Simultaneous boundedness of the operators $T^{(\lambda)}_a$ for all λ in the case of arbitrary (depending on both variables) nonnegative symbol was shown in [17]. We extend this result for a class of not necessarily nonnegative symbols depending only on y.

For a nonnegative function $a(t)$ we set

$$m_{a,0}(x) = \inf_{(0,x)} a(t/2) \quad \text{and} \quad m_{a,\infty}(x) = \inf_{(x/2,x)} a(t/2).$$

Corollary 4.10. *Given a nonnegative symbol $a(y)$, if either*

$$\lim_{x\to 0} m_{a,0}(x) = \infty \tag{4.12}$$

or

$$\lim_{x\to\infty} m_{a,\infty}(x) = \infty, \tag{4.13}$$

then the Toeplitz operator $T^{(\lambda)}_a$ is unbounded on each $\mathcal{A}^2_\lambda(\Pi)$, $\lambda \in [0, +\infty)$.

Proof. If the condition (4.12) holds then

$$B^{(1)}_{a,0}(\xi) = \int_0^{\xi} a(t/2)\mathrm{d}t \geqslant \xi m_{a,0}(\xi)$$

and $\xi^{-1}B^{(1)}(\xi) \to \infty \quad \text{as } \xi \to 0$. Now let the condition (4.13) holds. Then

$$\xi^{-1}B^{(1)}_{a,0}(\xi) > \xi^{-1}\int_{\xi/2}^{\xi} a(t/2)\mathrm{d}t \geqslant \frac{1}{2}m_{a,\infty}(\xi) \to \infty \quad \text{as } \xi \to \infty. \quad \blacksquare$$

Note that Corollary 4.10 shows that infinitely growing positive symbols cannot generate bounded Toeplitz operators. To generate a bounded Toeplitz operator, its unbounded symbol must necessarily have (see Example 4.4) a sufficiently sophisticated oscillating behavior at neighborhoods of the "critical" points 0 and ∞.

Given a symbol $a(y) \in L_1(\mathbb{R}_+, 0)$, denote by $B(a)$ the set of values $\lambda \in [0,\infty)$ for which the corresponding Toeplitz operator $T^{(\lambda)}_a$ is bounded. Theorem 4.6 suggests that the set $B(a)$, being nonempty, may have only one of the following three types:

$$[0,\infty), \quad [0,\nu), \quad [0,\nu].$$

We show that all of these possibilities can be realized. Indeed, the first case is satisfied for bounded symbols. The following theorem treats the two remaining cases.

Theorem 4.11. *There exists a family of symbols $a_{\nu,\beta}(y)$, with $\nu \in (0,1)$, $\beta \geqslant 0$, such that for the corresponding Toeplitz operators $T^{(\lambda)}_{a_{\nu,\beta}}$ we have:*
(i) *$B(a_{\nu,0}) = [0,\nu]$, $\beta = 0$;*
(ii) *$B(a_{\nu,\beta}) = [0,\nu)$, $\beta > 0$.*

Proof. To prove the above statement we show that the asymptotic behavior of the corresponding function $\gamma_{a_{\nu,\beta},\lambda}(x)$, when $x \to \infty$, is as follows,

$$\begin{aligned}\gamma_{a_{\nu,\beta},\lambda}(x) &= c_\lambda \mathrm{e}^{(\mathrm{i}/5\pi)\ln^2(1+x)} \ln^{\lambda-\nu}(1+x)\ln^{\beta}\ln(1+x) \\ &\quad + o(\ln^{\lambda-\nu}(1+x)\ln^{\beta}\ln(1+x)),\end{aligned} \tag{4.14}$$

where $c_\lambda \neq 0$, and

$$\lim_{x\to 0} \gamma_{a_{\nu,\beta},\lambda}(x) = 0. \tag{4.15}$$

To introduce the function $a_{\nu,\beta}(y)$ we consider

$$f_{\nu,\beta}(z) = \mathrm{e}^{((5\pi)/4)\mathrm{i}} \exp\Big\{\frac{\mathrm{i}}{5\pi}\ln^2(z+\mathrm{i})\Big\}\Big[\ln(z+\mathrm{i}) - \mathrm{i}\frac{5\pi}{2}\Big]^{-\nu} \ln^{\beta}\Big(\ln(z+\mathrm{i}) - \mathrm{i}\frac{5\pi}{2}\Big),$$

where the branch of the function $f_{\nu,\beta}(z)$ is fixed by imposing the condition $\arg z \in [3\pi/2, 7\pi/2]$. We set now

$$a_{\nu,\beta}(t/2) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f_{\nu,\beta}(x)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}x.$$

The function $f_{\nu,\beta}(z)$ belongs to the Hardy space $H^2(\Pi)$, hence $a_{\nu,\beta}(t) \in L_2(\mathbb{R}_+)$ and the formula

$$f_{\nu,\beta}(z) = \int\limits_{\mathbb{R}_+} a_{\nu,\beta}(t/2)\mathrm{e}^{\mathrm{i}zt}\mathrm{d}t$$

holds. Thus, $\gamma_{a_{\nu,\beta},0}(x) = xf_{\nu,\beta}(\mathrm{i}x)$.

Recall that $t^\alpha \mathrm{e}^{-xt} = \mathcal{D}^\alpha \mathrm{e}^{-xt}$, where the Liouville fractional derivative is given, as usually, as follows

$$\mathcal{D}^\alpha\varphi(x) = \frac{1}{d_{1,1}(\alpha)}\int\limits_{\mathbb{R}_+}\frac{\varphi(x+t)-\varphi(x)}{t^{1+\alpha}}\mathrm{d}t, \quad d_{1,1}(\alpha) = \int\limits_{\mathbb{R}_+}\frac{\mathrm{e}^{-\xi}-1}{\xi^{1+\alpha}}\mathrm{d}\xi, \quad 0<\alpha<1.$$

Therefore, denoting $c(\lambda) = 1/(d_{1,1}(\lambda)\Gamma(\lambda+1))$, we have

$$\begin{aligned}
\gamma_{a_{\nu,\beta},\lambda}(x) &= c(\lambda)x^{\lambda+1}\int\limits_{\mathbb{R}_+}\frac{f_{\nu,\beta}(\mathrm{i}(x+t)) - f_{\nu,\beta}(\mathrm{i}x)}{t^{1+\lambda}}\mathrm{d}t \\
&= c(\lambda)\frac{x^{\lambda+1}}{(x+1)^\lambda}\int\limits_{\mathbb{R}_+}\frac{f_{\nu,\beta}(\mathrm{i}(x+xt+t)) - f_{\nu,\beta}(\mathrm{i}x)}{t^{1+\lambda}}\mathrm{d}t \\
&= c(\lambda,x)(x+1)\int\limits_{\mathbb{R}_+}\frac{\mathrm{d}t}{t^{1+\lambda}}\mathrm{d}t\int\limits_0^t\frac{d}{\mathrm{d}\xi}f_{\nu,\beta}(\mathrm{i}(x+x\xi+\xi))\mathrm{d}\xi \\
&= c(\lambda,x)(x+1)\int\limits_{\mathbb{R}_+}\frac{\mathrm{d}}{\mathrm{d}\xi}f_{\nu,\beta}(\mathrm{i}(x+x\xi+\xi))\mathrm{d}\xi\int\limits_\xi^\infty\frac{\mathrm{d}t}{t^{1+\lambda}}\mathrm{d}t \\
&= \lambda^{-1}c(\lambda,x)(x+1)\int\limits_{\mathbb{R}_+}\frac{1}{\xi^\lambda}\frac{d}{\mathrm{d}\xi}f_{\nu,\beta}(\mathrm{i}(x+x\xi+\xi))\mathrm{d}\xi,
\end{aligned}$$

where $c(\lambda,x) = c(\lambda)x^{\lambda+1}/(1+x)^{\lambda+1}$. Note that

$$f_{\nu,\beta}(\mathrm{i}y) = \frac{\exp\{\frac{\mathrm{i}}{5\pi}\ln^2(1+y)\}}{1+y}\ln^{-\nu}(1+y)\ln^\beta\ln(1+y),$$

whence we have

$$\begin{aligned}
\gamma_{a_{\nu,\beta},\lambda}(x) = -\lambda^{-1}c(\lambda,x)\int\limits_{\mathbb{R}_+}&\frac{\exp\{\frac{\mathrm{i}}{5\pi}\ln^2(1+x)(1+\xi)\}}{\xi^\lambda(1+\xi)^2}\Big(\frac{2\mathrm{i}}{5\pi}\omega_{\nu,\beta}(x,\xi) \\
&- \omega_{\nu+1,\beta}(x,\xi) - \nu\omega_{\nu+2,\beta}(x,\xi) - \beta\omega_{\nu+2,\beta-1}(x,\xi)\Big)\mathrm{d}\xi,
\end{aligned} \tag{4.16}$$

where

$$\omega_{\nu,\beta}(x,\xi) = \ln^{1-\nu}(1+x)(1+\xi)\ln^\beta\ln(1+x)(1+\xi).$$

We split the above integral into four integrals according to the sum of four terms in the brackets. These integrals are of the same type, and differ (up to a constant) only by the parameters ν, β. Obviously, the principal term of the behavior of $\gamma_{a,\lambda}(x)$

when $x \to \infty$ is determined by the integral corresponding to the first summand, i.e.,

$$\begin{aligned} I(x,\lambda,\nu,\beta) &= \int_{\mathbb{R}_+} \frac{\exp\{\frac{\mathrm{i}}{5\pi}\ln^2(1+x)(1+\xi)\}}{\xi^\lambda(1+\xi)^2}\,\omega_{\nu,\beta}(x,\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}_+} \frac{\exp\{\frac{\mathrm{i}}{5\pi}\ln^2(1+x)(1+\xi)\}}{\xi^\lambda(1+\xi)^2}\,\omega_{\nu,\beta}(x,\xi)\,(\chi_0(\xi)+\chi_\infty(\xi))\mathrm{d}\xi \\ &= I_0(x,\lambda,\nu,\beta) + I_\infty(x,\lambda,\nu,\beta). \end{aligned}$$

Here $\chi_0(\xi)$ is a smooth function on $\mathbb{R}_+$, satisfying the conditions $\chi_0(\xi) = 1$ for $0 \leqslant \xi \leqslant 1$ and $\chi_0(\xi) = 0$ for $\xi \geqslant 2$; and $\chi_\infty(\xi) = 1 - \chi_0(\xi)$.

Integrating by parts the second integral we have

$$\begin{aligned} I_\infty(x,\lambda,\nu,\beta) &= -\frac{5\pi\mathrm{i}}{2}\int_1^\infty \frac{\omega_{\nu+1,\beta}(x,\xi)}{\xi^\lambda(1+\xi)}\,\chi_\infty(\xi)\,\mathrm{d}\exp\Big\{\frac{\mathrm{i}}{5\pi}\ln^2(1+x)(1+\xi)\Big\} \\ &= \frac{5\pi\mathrm{i}}{2}\int_1^\infty \exp\Big\{\frac{\mathrm{i}}{5\pi}\ln^2(1+x)(1+\xi)\Big\}\frac{\partial}{\partial\xi}\left(\frac{\omega_{\nu+1,\beta}(x,\xi)}{\xi^\lambda(1+\xi)}\,\chi_\infty(\xi)\right)\mathrm{d}\xi. \end{aligned}$$

For $\xi > 1$ and large enough x the following inequality holds

$$\left|\frac{\partial}{\partial\xi}\left(\frac{\omega_{\nu+1,\beta}(x,\xi)}{\xi^\lambda(1+\xi)}\,\chi_\infty(\xi)\right)\right| \leqslant \mathrm{const}\,\frac{\omega_{\nu+1,\beta}(x,0)}{\xi^\lambda(1+\xi)^2}.$$

Thus we have

$$|I_\infty(x,\lambda,\nu,\beta)| = O(\omega_{\nu+1,\beta}(x,0)) = O(\ln^{-\nu}(1+x)\,\ln^\beta\ln(1+x)).$$

For $I_0(x,\lambda,\nu,\beta)$ according to Lemma of Erdĺyi ([6]), we have

$$I_0(x,\lambda,\nu,\beta) = (1+O(\ln^{-1}(1+x)))\widetilde{I_0}(x,\lambda,\nu,\beta),$$

where

$$\begin{aligned} \widetilde{I_0}(x,\lambda,\nu,\beta) &= \omega_{\nu,\beta}(x,0)\mathrm{e}^{\mathrm{i}/(5\pi)\ln^2(1+x)}\int_{\mathbb{R}_+}\frac{\mathrm{e}^{\mathrm{i}(2\ln(1+x))/(5\pi)\,\xi}}{\xi^\lambda(1+\xi)^2}\chi_0(\xi)\mathrm{e}^{\mathrm{i}/(5\pi)\ln^2(1+\xi)}\mathrm{d}\xi \\ &= \omega_{\nu,\beta}(x,0)\mathrm{e}^{\mathrm{i}/(5\pi)\ln^2(1+x)}\int_0^2\frac{\mathrm{e}^{\mathrm{i}(2\ln(1+x))/(5\pi)\,\xi}}{\xi^\lambda}F(\xi)\mathrm{d}\xi. \end{aligned}$$

Applying Lemma of Erdĺyi ([6]) once again, we have

$$\begin{aligned} \int_0^2\frac{\mathrm{e}^{\mathrm{i}(2\ln(1+x))/(5\pi)\,\xi}}{\xi^\lambda}F(\xi)\mathrm{d}\xi &= \frac{5\pi}{2\mathrm{i}}\Gamma(1-\lambda)\mathrm{e}^{\mathrm{i}\pi(1-\lambda)/2}\mathrm{e}^{\mathrm{i}/(5\pi)\ln^2(1+x)}\ln^{\lambda-1}(1+x) \\ &\quad + o(\ln^{\lambda-1}(1+x)), \quad x\to\infty. \end{aligned}$$

This and the above considerations prove (4.14). Finally, it is easy to see that (4.16) implies (4.15). ∎

5. SPECTRA OF TOEPLITZ OPERATORS WITH SYMBOLS DEPENDING ON $y = \operatorname{Im} z$.

5.1. Continuous symbols. Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point (normally $E = (0, +\infty)$), and suppose that, for each $\lambda \in E$, we are given a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that:

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$;
(ii) $\lim\limits_{n\to\infty} \lambda_n = +\infty$;
(iii) $z = \lim\limits_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of a family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

For the case when E is a discrete set with a unique limit point at infinity, the above notion coincides with the partial limiting set introduced in [10], Section 3.1.1. Following the arguments of Proposition 3.5 in [10], one can show that

$$M_\infty = \bigcap_{\lambda} \operatorname{clos}\Big(\bigcup_{\mu\geqslant\lambda} M_\mu\Big).$$

Note that

$$\lim_{\lambda\to+\infty} M_\lambda = \lim_{\lambda\to+\infty} \overline{M}_\lambda = M_\infty.$$

The *a priori* spectral information for L_∞-symbols (see, for example, [3], [4]) says that for each $a \in L_\infty(\Pi)$ and each $\lambda \geqslant 0$

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\text{ess-Range}\, a). \tag{5.1}$$

Given a symbol $a = a(y)$, the Toeplitz operator $T_a^{(\lambda)}$ acting on the space $\mathcal{A}^2_\lambda(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where the function $\gamma_{a,\lambda}(x)$, $x \in \mathbb{R}_+$, is given by (2.6). Thus, we have obviously

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)},$$

where $M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}$.

Theorem 5.1. *Let $a = a(y) \in C(\overline{\mathbb{R}}_+) = C[0, +\infty]$. Then*

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \operatorname{Range} a. \tag{5.2}$$

Note that $\operatorname{Range} a$ coincides with the spectrum $\operatorname{sp} aI$ of the operator of multiplication by $a = a(y)$ acting, say, on all of $L_2(\Pi, \mathrm{d}\mu_\lambda)$, and hence another form of (5.2) is

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{sp} aI.$$

Proof. We use the Laplace method ([6]) to evaluate the integrals. Introduce the large parameter $L = \sqrt{x^2 + \lambda^2}$ (recall $\lambda \to +\infty$) and represent $\gamma_{a,\lambda}(x)$ in the form

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(t/2) \mathrm{e}^{-LS(t,\varphi)} \mathrm{d}t, \tag{5.3}$$

where

$$S(t,\varphi) = \frac{x}{L}\,t - \frac{\lambda}{L}\,\ln t = (\sin\varphi)\,t + (\cos\varphi)\,\ln\frac{1}{t}, \quad \text{with } \varphi \in \Big[0, \frac{\pi}{2}\Big].$$

The function $S(t,\varphi)$, as a function of t, has a minimum at the point

$$t_\varphi = \frac{\cos\varphi}{\sin\varphi} \in (0,\infty).$$

Write (5.3) in the form

$$\begin{aligned}\gamma_{a,\lambda}(x) - a(t_\varphi/2) &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\Big[\int\limits_{\mathbb{R}_+\cap U(t_\varphi)} (a(t/2)-a(t_\varphi/2))\mathrm{e}^{-LS(t,\varphi)}\mathrm{d}t \\ &\qquad + \int\limits_{\mathbb{R}_+\setminus U(t_\varphi)} (a(t/2)-a(t_\varphi/2))\mathrm{e}^{-LS(t,\varphi)}\mathrm{d}t\Big] \\ &\equiv I_1(L) + I_2(L),\end{aligned} \tag{5.4}$$

where $U(t_\varphi)$ is a neighborhood of the point t_φ such that $\sup\limits_{t\in U(t_\varphi)} |a(t/2)-a(t_\varphi/2)| < \varepsilon$, with $\varepsilon > 0$ sufficiently small. We have,

$$I_1(L) \leqslant \varepsilon$$

uniformly in φ. Next, $I_2(L) \leqslant \varepsilon$ uniformly on φ as well. Indeed, rewrite the integral $I_2(L)$ in the following form

$$\begin{aligned}I_2(L) &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int\limits_0^{t_{\varphi-\sigma}} (a(t/2)-a(t_\varphi/2))\mathrm{e}^{-LS(t,\varphi)}\mathrm{d}t \\ &\quad + \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int\limits_{t_\varphi+\sigma}^{\infty} (a(t/2)-a(t_\varphi/2))\mathrm{e}^{-LS(t,\varphi)}\mathrm{d}t \\ &\equiv I_{2,1}(L) + I_{2,2}(L)\end{aligned}$$

where $\sigma > 0$ is small enough.

Use the asymptotic Euler formula for the Γ-function (see, formula 8.327 of [7])

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda) = \frac{\lambda\mathrm{e}^{-\lambda}\lambda^{\lambda-1/2}}{\sqrt{2\pi}}(1+O(\lambda^{-1/2})), \quad \lambda\to\infty,$$

where we set $\lambda = xt_\varphi$. Then the integral $I_{2,2}(L)$ admits the following estimate

$$|I_{2,2}(L)| \leqslant \mathrm{const}\, x^{1/2} \int\limits_{t_\varphi+\sigma}^{\infty} |a(t/2)-a(t_\varphi/2)|\mathrm{e}^{-x\widetilde{S}(t,\varphi)}\mathrm{d}t$$

where

$$\widetilde{S}(t,\varphi) = (t-t_\varphi) - t_\varphi(\ln t - \ln t_\varphi).$$

It is evident that there exists Δ (>0) which does not depend on φ and such that, for $t \geqslant t_\varphi + \delta$, the following inequality holds

$$\widetilde{S}(t,\varphi) > \Delta(t - t_\varphi), \quad t > t_\varphi.$$

Thus, we have

$$\begin{aligned} |I_{2,2}(L)| &\leqslant \operatorname{const} x^{1/2} \int\limits_{t_{\varphi+\sigma}} |a(t/2) - a(t_\varphi/2)| \mathrm{e}^{-x\Delta(t-t_\varphi)} \mathrm{d}t \\ &\leqslant \operatorname{const} x^{1/2} \mathrm{e}^{-(x-1)\Delta\sigma} \int\limits_{t_\varphi+\sigma} |a(t/2) - a(t_\varphi/2)| \mathrm{e}^{-\Delta(t-t_\varphi)} \mathrm{d}t. \end{aligned}$$

According to the definition of the class $L_1(\mathbb{R}_+, 0)$ the last integral is finite and we have, uniformly on φ,

$$\lim_{L\to\infty} I_{2,2}(L) = 0.$$

Analogously one can get that, uniformly with respect to φ,

$$\lim_{L\to\infty} I_{2,1}(L) = 0$$

and, consequently, $\lim\limits_{L\to\infty} I_2(L) = 0$.

Since ε can be arbitrarily small, from the above we get

$$\gamma_{a,\lambda}(x) = a(t_\varphi/2)\ (1 + \alpha(L)), \tag{5.5}$$

where $\alpha(L) \to 0$, when $L \to \infty$, uniformly with respect to φ. ∎

5.2. PIECEWISE CONTINUOUS SYMBOLS. Let $b(t) = a(t/2)$ be a piecewise continuous function on $[0, +\infty]$ having jumps on a finite set of points $\{t_j\}_{j=1}^m$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = +\infty,$$

and $a(t_j/2 \pm 0), j = 1, \ldots, m$, exist. Introduce the sets

$$J_j(a) := \{z \in \mathbb{C} : z = a(t/2), t \in (t_j, t_{j+1})\}$$

where $j = 0, \ldots, m$, and let $I_j(a)$ be the straight line segment with the endpoints $a(t_j/2 - 0)$ and $a(t_j/2 + 0)$, $j = 1, 2, \ldots, m$.

Introduce now

$$\widetilde{R}(a) = \Big(\bigcup_{j=0}^{m} J_j(a)\Big) \cup \Big(\bigcup_{j=1}^{m} I_j(a)\Big).$$

THEOREM 5.2. *Let $a(t/2)$ be a piecewise continuous function on* $[0, +\infty]$. *Then*

$$\lim_{\lambda\to\infty} \operatorname{sp}_\lambda T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

Proof. The proof is quite analogous to that one of Theorem 5.2 in [9]; see also [8]. ∎

For L_∞-symbols, apart from the *a priori* information (5.1), we have obviously

$$\lim_{\lambda\to\infty} \mathrm{sp}_\lambda T_a^{(\lambda)} = M_\infty(a) \subset \mathrm{conv}(\mathrm{ess\,Range}\, a). \tag{5.6}$$

At the same time the collocation of $M_\infty(a)$ inside conv(ess Range a) may essentially vary. We give a number of examples illustrating possible interrelations between these sets.

EXAMPLE 5.3. Let $a(t) \in C[0,+\infty]$. Then, according to Theorem 5.1,

$$M_\infty(a) = \mathrm{Range}\, a \ (= \mathrm{ess\,Range}\, a).$$

EXAMPLE 5.4. Let

$$a(t/2) = \begin{cases} \alpha_1 & t \in (0,1); \\ \alpha_2 & t \in [1,\infty]. \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1 \neq \alpha_2$. Then, according to Theorem 5.2, $M_\infty(a)$ coincides with the straight line segment $[\alpha_1, \alpha_2]$ joining the points α_1 and α_2, whence

$$M_\infty(a) = \mathrm{conv}(\mathrm{ess\,Range}\, a) \quad (= \mathrm{conv}(\mathrm{Range}\, a)).$$

EXAMPLE 5.5. Let

$$a(t/2) = \begin{cases} \alpha_1 & t \in [0,1), \\ \alpha_2 & t \in [1,2), \\ \alpha_3 & t \in [2,\infty]; \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are different points from $\mathbb{C}$. Then, by Theorem 5.2, we have

$$M_\infty(a) = [\alpha_1, \alpha_2] \cup [\alpha_2, \alpha_3]$$

and in this case the set $M_\infty(a)$ is a part of the boundary of the convex hull ess Range a = Range a, that is

$$M_\infty(a) \subset \partial\, \mathrm{conv}(\mathrm{Range}\, a).$$

EXAMPLE 5.6. Let $\alpha_1, \alpha_2, \alpha_3$ be as above, and

$$a(t/2) = \begin{cases} \alpha_1 & t \in [0,1), \\ \alpha_2 & t \in [1,2), \\ \alpha_3 & t \in [2,3), \\ \alpha_1 & t \in [3,\infty]. \end{cases}$$

By Theorem 5.2 the set $M_\infty(a)$ coincides with triangle with the vertices $\alpha_1, \alpha_2, \alpha_3$,

$$M_\infty(a) = [\alpha_1, \alpha_2] \cup [\alpha_2, \alpha_3] \cup [\alpha_3, \alpha_4].$$

Thus, in this case,

$$M_\infty(a) = \partial\, \mathrm{conv}(\mathrm{Range}\, a).$$

EXAMPLE 5.7. Let $\{t_j\}_{j\in\mathbb{Z}_+}$ be an increasing sequence of positive numbers with $\lim\limits_{j\to\infty} t_j = \infty$ and $t_0 = 0$. Define the symbol $a(t)$ as follows,

$$a(t/2) = \begin{cases} e^{i\xi_j} & t \in [t_{2j}, t_{2j+1}), \\ -e^{i\xi_j} & t \in [t_{2j+1}, t_{2j+2}), \end{cases}$$

where $\{\xi_j\}_{j\in\mathbb{Z}_+} \subset [0,\pi]$ with the closure $\overline{\{\xi_j\}_{j\in\mathbb{Z}_+}} = [0,\pi]$.

As in Theorem 5.2 one can show that each diameter $[e^{i\xi_j}, -e^{i\xi_i}]$ of the unit disk $\mathbb{D}$ having $e^{i\xi_j}$ and $-e^{i\xi_j}$ as endpoints, belongs to $M_\infty(a)$, which implies $\overline{\mathbb{D}} \subset M_\infty(a)$. We have that $\overline{\operatorname{Range} a} = \partial\mathbb{D} = \mathbb{T}$. Finally,

$$M_\infty(a) = \overline{\mathbb{D}} = \operatorname{conv}(\operatorname{Range} a).$$

5.3. OSCILLATING SYMBOLS. We consider here the case of a discontinuity of the second kind, the oscillating symbols. To be more precise, the following two model situations will be considered: a strong oscillation and a slow oscillation. In spite of their qualitative identity, an oscillation type discontinuity, the results differ drastically.

THEOREM 5.8. (Strong oscillation) *Let* $a(t) = e^{2it}$, *then* $\operatorname{Range} a = \mathbb{T}$ *and* $M_\infty(a) = \mathbb{D}$.

Proof. For $a(t/2) = e^{it}$ we have

$$\begin{aligned}\gamma_{a,\lambda}(x) &= \frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty t^\lambda e^{-(x-i)t}dt \\ &= \frac{x^{\lambda+1}}{(x-i)^{\lambda+1}} \cdot \frac{1}{\Gamma(\lambda+1)}\int_0^\infty s^\lambda e^{-s}ds \\ &= \left(\frac{x}{x-i}\right)^{\lambda+1} \\ &= \exp\left[\frac{\lambda+1}{2}\ln\left(1-\frac{1}{x^2+1}\right)\right]\cdot\exp\left[(\lambda+1)i\arctan(x^{-1})\right].\end{aligned} \tag{5.7}$$

Given a nonzero point $z_0 \in \mathbb{D}$, we represent it in the following form

$$z_0 = \exp(-\alpha_0 + i\beta_0),$$

where $\alpha_0 > 0$ and $\beta_0 \in [0, 2\pi)$.

Introduce the sequences

$$x_k = \frac{\beta_0 + 2\pi k}{2\alpha_0} \quad \text{and} \quad \lambda_k = \frac{(\beta_0 + 2\pi k)^2}{2\alpha_0} - 1 = 2\alpha_0 x_k^2 - 1, \quad k \in \mathbb{N}.$$

Then, for large values of k, we have

$$\begin{aligned}\gamma_{a,\lambda_k}(x_k) &= \exp\left[\frac{\lambda_k+1}{2}\ln\left(1-\frac{1}{x_k^2+1}\right)\right]\cdot\exp\left[(\lambda_k+1)i\arctan(x_k^{-1})\right] \\ &= \exp\left[-\frac{\lambda_k+1}{2x_k^2} + (\lambda_k+1)O(x_k^{-4})\right] \\ &\quad \times \exp\left[i\frac{\lambda_k+1}{x_k} + (\lambda_k+1)O(x_k^{-3})\right] \\ &= \exp\left[-\alpha_0 + O(k^{-2}) + i(\beta_0 + 2\pi k) + O(k^{-1})\right].\end{aligned}$$

It is easy to see now that

$$\lim_{k\to\infty}\gamma_{a,\lambda_k}(x_k) = z_0,$$

that is, $z_0 \in M_\infty(a)$, and $\mathbb{D} \subset M_\infty(a)$. The inverse inclusion follows from (5.6). ∎

We note that formula (5.7) permits us to understand the form of the image of $\gamma_{a,\lambda}$ for each fixed (and sufficiently large) value of λ. First of all, it is easy to see that

$$\lim_{x\to\infty}\gamma_{a,\lambda}(x)=1 \quad\text{and}\quad \lim_{x\to 0}\gamma_{a,\lambda}(x)=0.$$

If $0<m<x<M<+\infty$, then the absolute value of $\gamma_{a,\lambda}(x)$ changes much more slowly than its argument. That is, for each fixed λ, the image of $\gamma_{a,\lambda}$ looks like a spiral outgoing from the point $z=1$ and tending to $z=0$, as x tends to 0. Moreover, when λ is growing, the branches of a spiral became closer and closer to each other.

THEOREM 5.9. (Slow oscillation) *Let* $a(t)=(2t)^{\mathrm{i}}$, *then* $\operatorname{Range} a=\mathbb{T}$ *and* $M_\infty(a)=\mathbb{T}$.

Proof. For $a(t/2)=t^{\mathrm{i}}$ we have

$$\begin{aligned}\gamma_{a,\lambda}(x)&=\frac{x^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty t^{\lambda+\mathrm{i}}\mathrm{e}^{-xt}\mathrm{d}t\\&=\frac{1}{\Gamma(\lambda+1)}\int_0^\infty s^{\lambda+\mathrm{i}}\mathrm{e}^{-s}\mathrm{d}s=x^{\mathrm{i}}\,\frac{\Gamma(\lambda+1+\mathrm{i})}{\Gamma(\lambda+1)}.\end{aligned}$$

That is for a fixed λ the image of $\gamma_{a,\lambda}$ coincides with the circle centered at origin and having radius equals to $|(\Gamma(\lambda+1+\mathrm{i}))/(\Gamma(\lambda+1))|$.

By formula 8.328.2 of [7], we have

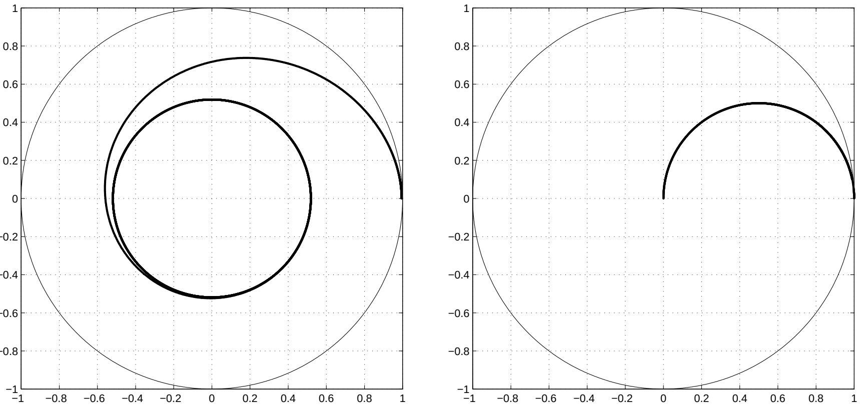
$$\lim_{\lambda\to\infty}\left|\frac{\Gamma(\lambda+1+\mathrm{i})}{\Gamma(\lambda+1)}\right|=1. \quad\blacksquare$$

We note that Theorems 5.8 and 5.9 can be generalized for a wide class of strong and slowly oscillating symbols. For example, if $a_1(t)=(2t+1)^{\mathrm{i}}$, then $M_\infty(a_1)=\mathbb{T}$, as in Theorem 5.9. The function $a_1(t)$ is continuous at the point $t=0$, thus $\gamma_{a_1,\lambda}(\infty)=a_1(0)=1$, for all λ. For a fixed λ the image of $\gamma_{a_1,\lambda}$ is a spiral outgoing from the point $z=1$ and tending to the limit circle with the radius equals to $|(\Gamma(\lambda+1+\mathrm{i}))/(\Gamma(\lambda+1))|$ and centered at origin (the same circle as in Theorem 5.9).

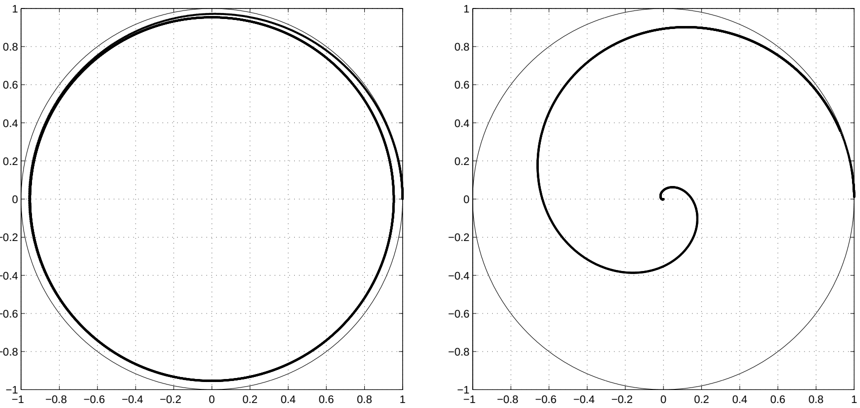
We illustrate the above on the figures presenting the images of functions $\gamma_{a,\lambda}$ for two oscillating symbols

$$a_1(t)=(1+2t)^{\mathrm{i}}=\mathrm{e}^{\mathrm{i}\ln(1+2t)} \quad\text{and}\quad a_2(t)=\mathrm{e}^{\mathrm{i}2t}, \quad t\in[0,\infty),$$

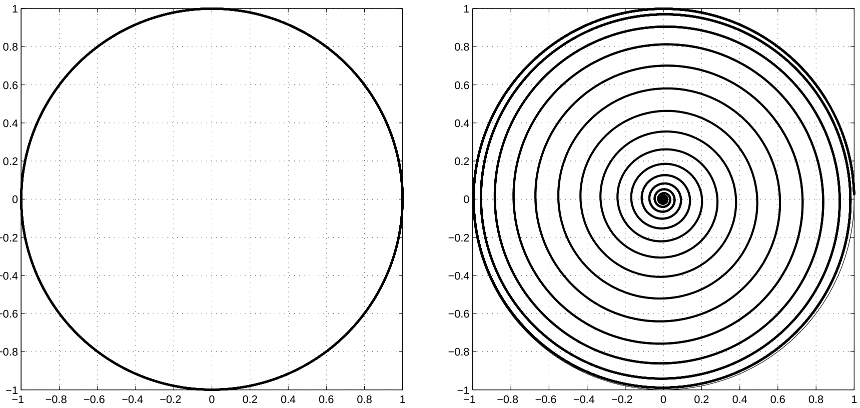
and for the following values of λ: 0, 10, and 1000.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 0$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 10$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 1000$.

We note that both symbols are continuous at the point $t = 0$ and have an oscillation type discontinuity at infinity, both of them are of the same form

$$a_k(t) = \mathrm{e}^{\mathrm{i}\varphi_k(t)}, \quad k = 1, 2,$$

where the corresponding functions $\varphi_k(t)$ are continuous and growing on $[0, +\infty]$ with $\varphi_k(0) = 0$ and $\varphi_k(+\infty) = +\infty$. The only difference between them is the speed of their growth at infinity. And this difference leads to a drastic difference between the spectral behavior of the corresponding Toeplitz operators.

5.4. UNBOUNDED SYMBOLS.

THEOREM 5.10. *Let $a(t) \in L_1(\mathbb{R}_+, 0) \cap C(\mathbb{R}_+)$. Then*

$$\operatorname{Range} a \subset M_\infty(a).$$

Proof. The proof is analogous to that of Theorem 5.1. ∎

We show now that the property (5.6), previously established for bounded symbols, still remains valid for our unbounded symbols.

THEOREM 5.11. *Let $a(t) \in L_1(\mathbb{R}_+, 0)$. Then*

$$M_\infty(a) \subset \operatorname{conv}(\operatorname{ess\,Range} a).$$

Proof. For each $M > 0$ consider the function

$$a_M(t) = \begin{cases} a(t) & \text{if } |a(t)| \leqslant M, \\ 0 & \text{if } |a(t)| > M. \end{cases}$$

The function $a_M(t)$ is bounded, whence

$$\operatorname{Range} \gamma_{a_M,\lambda} \subset \operatorname{conv}(\operatorname{ess\,Range} a_M) \subset \operatorname{conv}(\operatorname{ess\,Range} a).$$

The equality

$$\lim_{M \to \infty} \gamma_{a_M,\lambda}(x) = \gamma_{a,\lambda}(x),$$

verified by the Lebesgue dominated convergence theorem, implies

$$\operatorname{Range} \gamma_{a,\lambda}(x) \subset \operatorname{conv}(\operatorname{Range} a). \quad ∎$$

COROLLARY 5.12. *For functions $a(t) \in L_1(\mathbb{R}_+, 0) \cap C(\mathbb{R}_+)$,*

$$\operatorname{Range} a \subset M_\infty(a) \subset \operatorname{conv}(\operatorname{Range} a).$$

EXAMPLE 5.13. For each $j \in \mathbb{N}$ define $I_j = [j-1, j-1+1/j^3]$ and let $\{\xi_j\}_{j\in\mathbb{N}}$ be a sequence such that $\overline{\{\xi_j\}_{j\in\mathbb{N}}} = [0, 2\pi]$. Define the symbol as follows

$$a(t/2) = \begin{cases} j\mathrm{e}^{\mathrm{i}\xi_j} & t \in I_j, j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $B_a^{(1)}(\xi) \leqslant \sum_{j\in\mathbb{N}} 1/j^2$, and the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded for every $\lambda > 0$. Theorem 5.2 implies that the straight line segment $[0, j\mathrm{e}^{\mathrm{i}\xi_j}]$ is contained in $M_\infty(a)$. Thus

$$M_\infty(a) = \mathbb{C} = \operatorname{conv}(\operatorname{Range} a).$$

EXAMPLE 5.14. For given $\alpha \in (0,1)$ introduce $a(t/2) = t^{\mathrm{i}-\alpha}$ and calculate

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty t^{\lambda+\mathrm{i}-\alpha} \mathrm{e}^{-xt} \mathrm{d}t = \frac{x^{\alpha-\mathrm{i}}\Gamma(\lambda+1+\alpha-\mathrm{i})}{\Gamma(\lambda+1)}.$$

By the asymptotic of the Γ-function (see formula 8.327 in [7])

$$\gamma_{a,\lambda}(x) = x^{\alpha-\mathrm{i}}(\lambda+1)^{\mathrm{i}-\alpha}(1+o(1)), \quad \lambda \to \infty.$$

Given arbitrary $\eta > 0$, one can take x and λ such that $(\lambda+1)/x = \eta$. Thus,

$$\gamma_{a,\lambda}(x) = \eta^{\mathrm{i}-\alpha}(1+o(1)), \quad \lambda \to \infty,$$

and in this case,

$$\operatorname{Range} a = M_\infty(a).$$

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